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APPLIED REGRESSION IN THE PRESENCE OF X ERROR

Fourth Interim Report

Project 7788

Development of Statistical Procedures

Prepared By

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The New Jersey Department of Transportation
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16. Abstract

Regression analysis is frequently used in the engineering field to develop mathematical models for a wide variety of applications. Of the several assumptions upon which regression is based, one of the most fundamental is that the X values are known exactly and that any error is associated only with the Y measurements. Since this is not the case for many engineering applications, a study was conducted to (a) determine the magnitude of this problem and to (b) develop and test a software package that incorporates a theoretical solution found in the literature. Computer simulation is used to demonstrate both the seriousness of the problem and the effectiveness of the solution. An example based on early-strength tests of concrete demonstrates the use of the new software. The complete Fortran coding for this program is contained in the Appendix.

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1 INTRODUCTION

Many engineering applications require the development of a mathematical model (equation) to characterize some physical relationship. Examples include those such as the following:

CHARACTERISTIC OF INTEREST	X DATA (Independent Variable)	Y DATA (Dependent Variable)
Compressive strength of concrete	7-day test results	28-day test results
Rating of highway pavement serviceability	Output of mechanical roughness-measuring device	Average rating of a team of panelists
Rating of highway pavement serviceability	Cumulative axle loads	Current rating of serviceability

In the first example, the objective is a reliable early predictor of the 28-day strength of concrete, a measure upon which many acceptance procedures are based. The objective of the second example is to replace a costly and time consuming subjective rating procedure with a simple mechanical device. In the third example, a relationship is sought that will become an integral part of a pavement management system.

The variable to be predicted or estimated is placed on the Y axis and an equation of the form Y = F(X) is desired. The equation may be linear, quadratic, exponential, or any other appropriate form. The analyst, from his understanding of the physical process, will often know the correct form in advance. In other cases, it may be necessary to let the data dictate the form.

The desired relationship is often derived empirically from a set of X, Y data values using the technique of least squares (1) as illustrated in Figure 1. The procedure, invisible to the analyst when executed by a

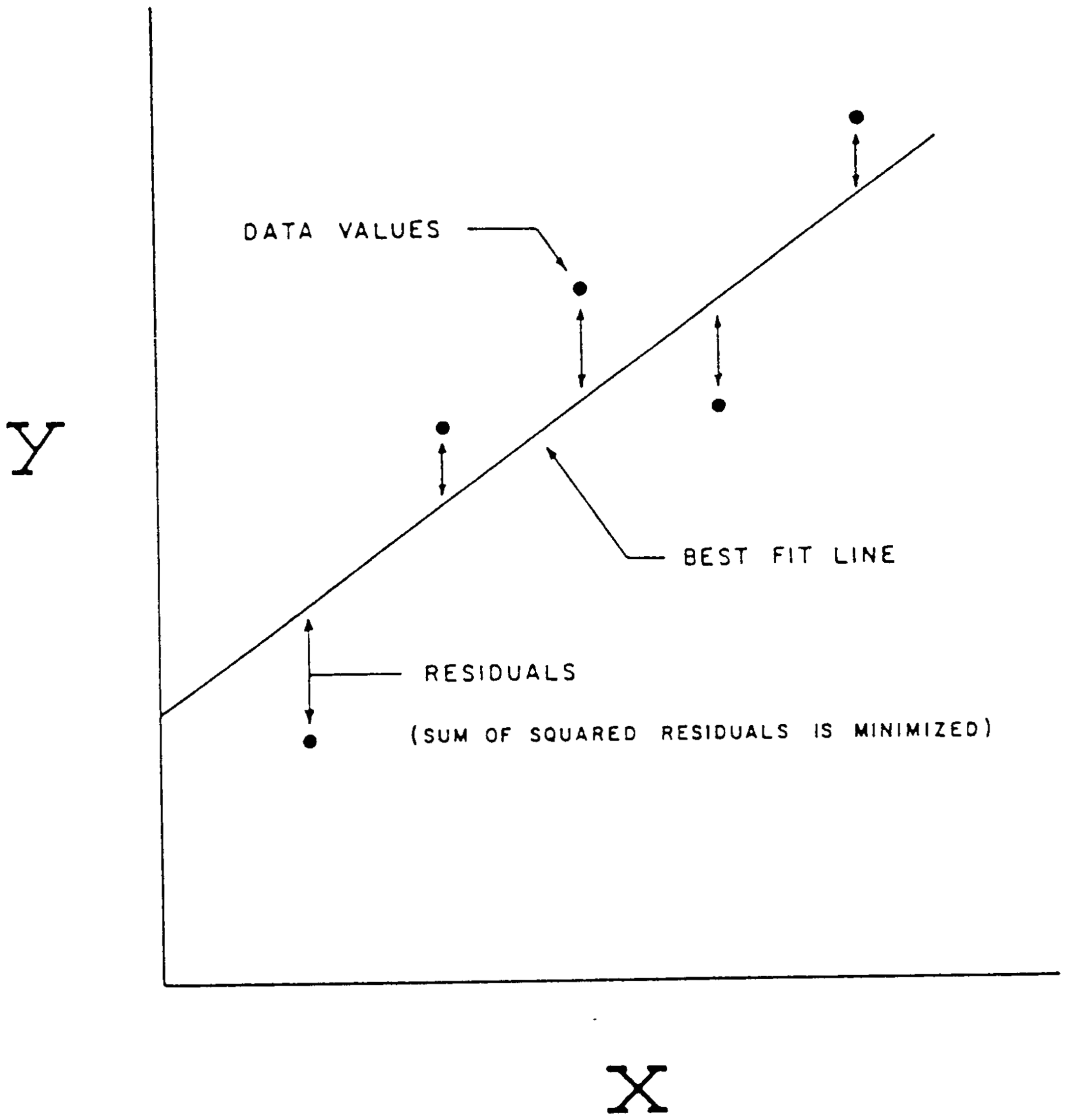


Figure 1. Conceptual illustration of the ordinary least squares technique.

computer program, consists of solving for the line that "best" fits the data. When ordinary least squares is used, the best fit is defined as that line of the chosen form that minimizes the sum of the squared residuals in a direction parallel to the Y axis.

In carrying out this procedure, several theoretical assumptions are made, one of the most fundamental of which is that the X values are known exactly and that any error of measurement is associated only with the Y values. Because it is often impossible or impractical to achieve this idealized condition in practice, this study was undertaken (a) to investigate the effect of failing to satisfy this assumption and (b) to develop and test a software package that incorporates a theoretical procedure for dealing with X error (2). This report addresses linear models since only the linear solution of the X-error problem has been published to date.

2 USE OF COMPUTER SIMULATION

In order to demonstrate the extent of the X-error problem and the effectiveness of the solution, a method was required to observe and quantify the accuracy and precision of the regression estimates. This can readily be accomplished with computer simulation by performing the following steps:

1. Randomly generate a bivariate normal X, Y data set with known regression (population) parameters:
    (a) intercept (β_0)
    (b) slope (β_1)
    (c) residual error (σ_{yx})

2. Include a fixed amount of X error, either in absolute terms or as a percentage of σ_{yx}.
3. Use the randomly generated data to estimate the regression parameters:
    - (a) intercept (B_0)
    - (b) slope (B_1)
    - (c) residual error (S_{yx})
4. Repeat the entire process many times in order to compare the distributions of the regression estimates with the known parameters. Ideally, the sampling distributions of the estimates should be centered on the true population parameters and have relatively narrow dispersions.

This technique can be used to provide a very dramatic demonstration of the bias introduced by error in the X variable and the conditions that accentuate it. It will also be used to demonstrate the effectiveness of the procedure developed to overcome this problem.

3 DEMONSTRATION OF THE PROBLEM

Table 1 has been prepared using dimensionless data to illustrate the detrimental effect even a moderate amount of X error can have under certain conditions. It can be observed that, when there is no X error, the estimated values (averages for 1000 replications) of all three regression parameters are extremely close to the true population values. When the amount of X error is as little as 25 percent of the Y error (σ_{yx}), it can be seen that a substantial amount of bias has been introduced in the estimates of both the intercept and the residual error. As the degree of X error increases, all three regression parameters begin to show considerable

Table 1. Examples of bias introduced by the presence of X error.

PARAMETER	TRUE VALUE	REGRESSION ESTIMATES (a) OBTAINED BY ORDINARY LEAST SQUARES FOR SELECTED LEVELS OF X ERROR (b)				
		0	25	50	75	100
Intercept	100	100.14	108.11	129.25	160.81	200.23
Slope	10	10.00	9.84	9.41	8.78	8.00
Residual Error	5	4.93	13.21	24.62	35.08	44.79

(a) Results obtained by computer simulation with 1000 replications of 30 data points spanning the range between approximately X = 30 and X = 70.

(b) X error is measured as the ratio σ_{XX}/σ_{YX}, expressed as a percent, in which σ_{XX} and σ_{YX} represent the precision with which the X and Y variables are measured.

bias. When the X error and the Y error are approximately equal, a fairly common situation in actual practice, the regression estimates differ substantially from the true population parameters.

It should be noted that this example was chosen to dramatize the potentially serious nature of the X-error problem. Although the three population parameters ($\beta_0 = 100$, $\beta_1 = 10$, $\sigma_{yx} = 5$) are not extreme in any sense, the effect is pronounced because the slope is fairly steep. Figure 2 presents a conceptual illustration of the effect of the slope in translating X error into apparent Y error, error that the ordinary least squares procedure attributes solely to the Y variable. The examples in Table 2 further demonstrate this effect. Viewed collectively, the examples in Tables 1 and 2 provide empirical evidence of two conditions that may influence the magnitude of the X-error problem, i.e., the ratio of X error to Y error and the slope of the regression line.

4 MANDEL'S SOLUTION

A theoretically derived procedure for avoiding the bias in the regression estimates due to X error has been published by Mandel (2). Use of the procedure requires one additional bit of information that is usually readily available or readily obtainable: the ratio of the variances associated with the X and Y measurements. Although the mathematical procedure is somewhat involved, the concept is very easy to visualize. Ordinary least squares minimizes the sum of the squared residuals in a direction parallel to the Y axis. As shown in Figure 3, Mandel's procedure performs the minimization process in a direction oblique to the X and Y axes, the exact angle being determined primarily by the relative magnitude of the X and Y error.

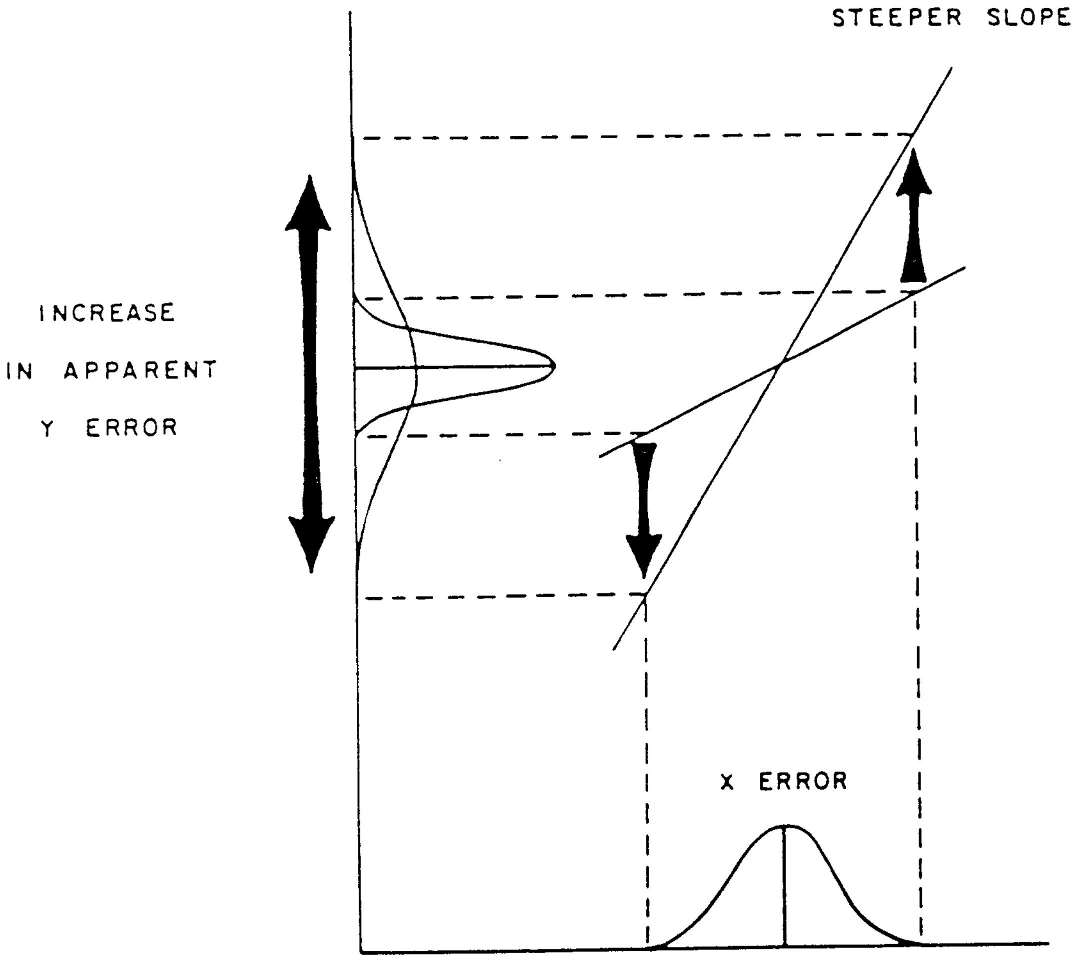


Figure 2. Effect of slope on the translation of X error into apparent Y error.

Table 2. Effect of slope on the degree of bias introduced.

PARAMETER	TRUE VALUE	REGRESSION ESTIMATES (a) OBTAINED BY ORDINARY LEAST SQUARES FOR FIXED X ERROR (b) AND SELECTED LEVELS OF SLOPE				
		0	25	50	75	100
Intercept	100	99.94	105.14	109.95	150.26	200.23
Slope	(c)	0.00	0.40	0.80	3.99	8.00
Residual Error	5	4.95	5.44	6.69	22.58	44.79

(a) Results obtained by computer simulation with 1000 replications of 30 data points spanning the range between approximately X = 30 and X = 70.

(b) The level of X error is fixed at 100 percent ($\sigma_{XX} = \sigma_{YX}$, where σ_{XX} and σ_{YX} represent the precision with which the X and Y variables are measured).

(c) Variable, values given in column headings.

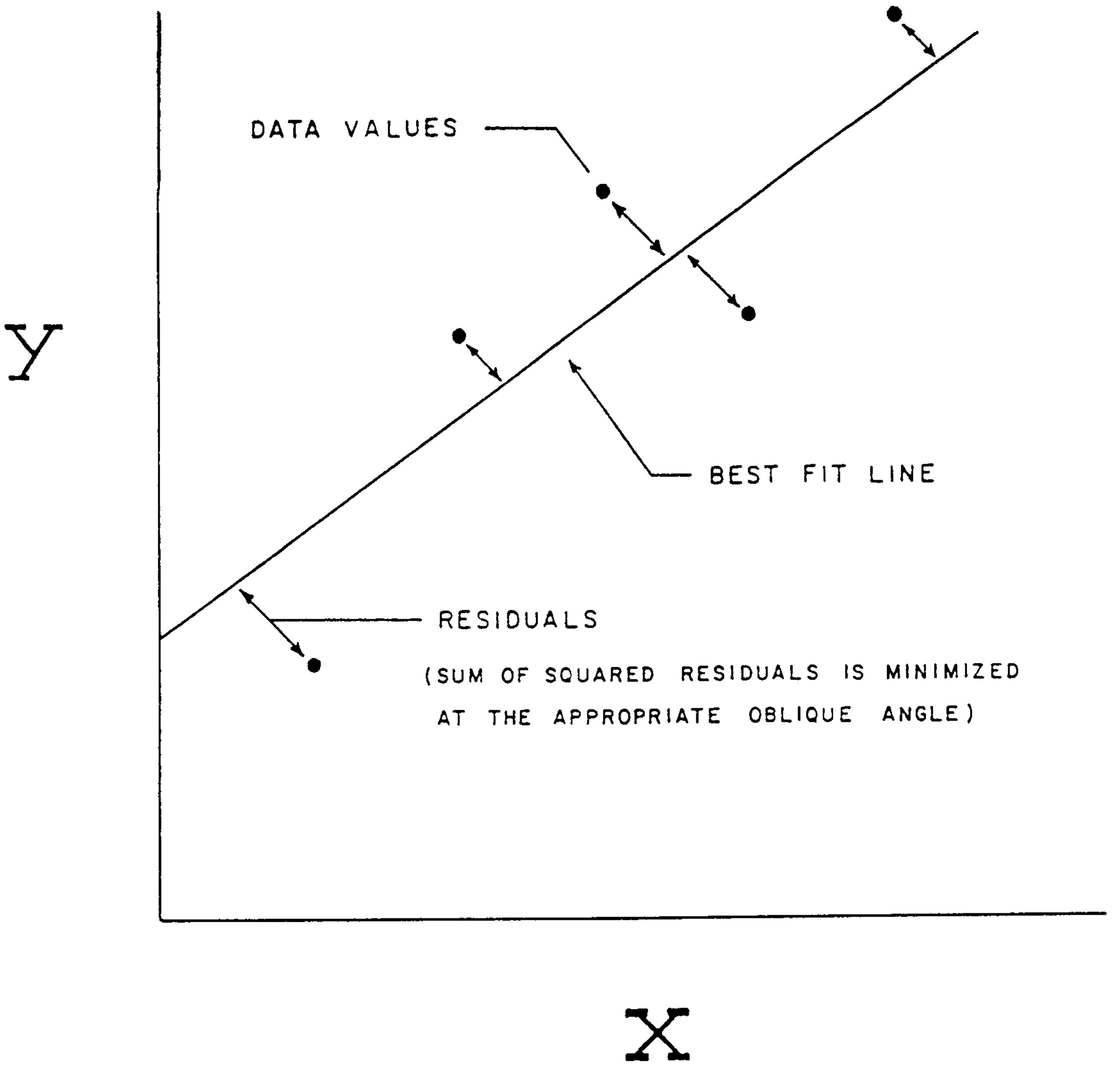


Figure 3. Conceptual illustration of Mandel's method.

In order to test the effectiveness of Mandel's method, it was applied to the same data sets used to develop Table 1. The results are reported in Table 3 and the values from Table 1 are repeated for ease of comparison. It can be seen from Table 3 that Mandel's method is extremely effective in removing the bias that exists when an application with X error is analyzed by ordinary least squares. Its only discernible weakness in this example is a possible small downward bias of the estimate of the intercept when the X error is quite large.

To judge whether or not this apparent bias was real, the simulation program was modified to print out a histogram and elementary statistics for 1000 intercept estimates. The X error was held constant at 100 percent of σ_{yx}. Although not strictly applicable because the distribution of intercept estimates was somewhat skewed, a t test indicated that the average intercept of 93.06 was highly significantly different ($\alpha < 0.001$) from the true value of 100.0. Although it is not obvious from the results in Table 3, a similar test suggests that the slope estimates may also be biased to a very small degree. Consequently, while Mandel's method appears to be very effective, and is far superior to ordinary least squares, it may not be not totally unbiased in all cases.

Figure 4 illustrates in a graphical way the effects that have been observed in Table 3. The distributions shown in this figure were drawn from histograms generated by the same simulation programs used to develop Tables 1 - 3. Mandel's method can be seen to be essentially unbiased in that the means of the distributions generated by that method are very close to the true population parameters. In marked contrast, the distributions produced by ordinary least squares are shifted substantially away from the true parameters. Another important observation in Figure 4 is that the

Table 3. Examples of bias introduced by the presence of X error.

PARAMETER	TRUE VALUE	METHOD	REGRESSION ESTIMATES (a) OBTAINED FOR SELECTED LEVELS OF X ERROR (b)				
			0	25	50	75	100
Intercept	100	Mandel	100.18	100.19	98.34	96.69	93.06
		OLS (c)	100.14	108.11	129.25	160.81	200.23
Slope	10	Mandel	10.00	9.99	10.03	10.06	10.14
		OLS (c)	10.00	9.84	9.41	8.78	8.00
Residual Error	5	Mandel	4.93	4.94	4.97	4.94	4.97
		OLS (c)	4.93	13.21	24.62	35.08	44.79

(a) Results obtained by computer simulation with 1000 replications of 30 data points spanning the range between approximately X = 30 and X = 70.

(b) X error is measured as the ratio σ_{XX}/σ_{YX}, expressed as a percent, in which σ_{XX} and σ_{YX} represent the precision with which the X and Y variables are measured.

(c) Ordinary least squares.

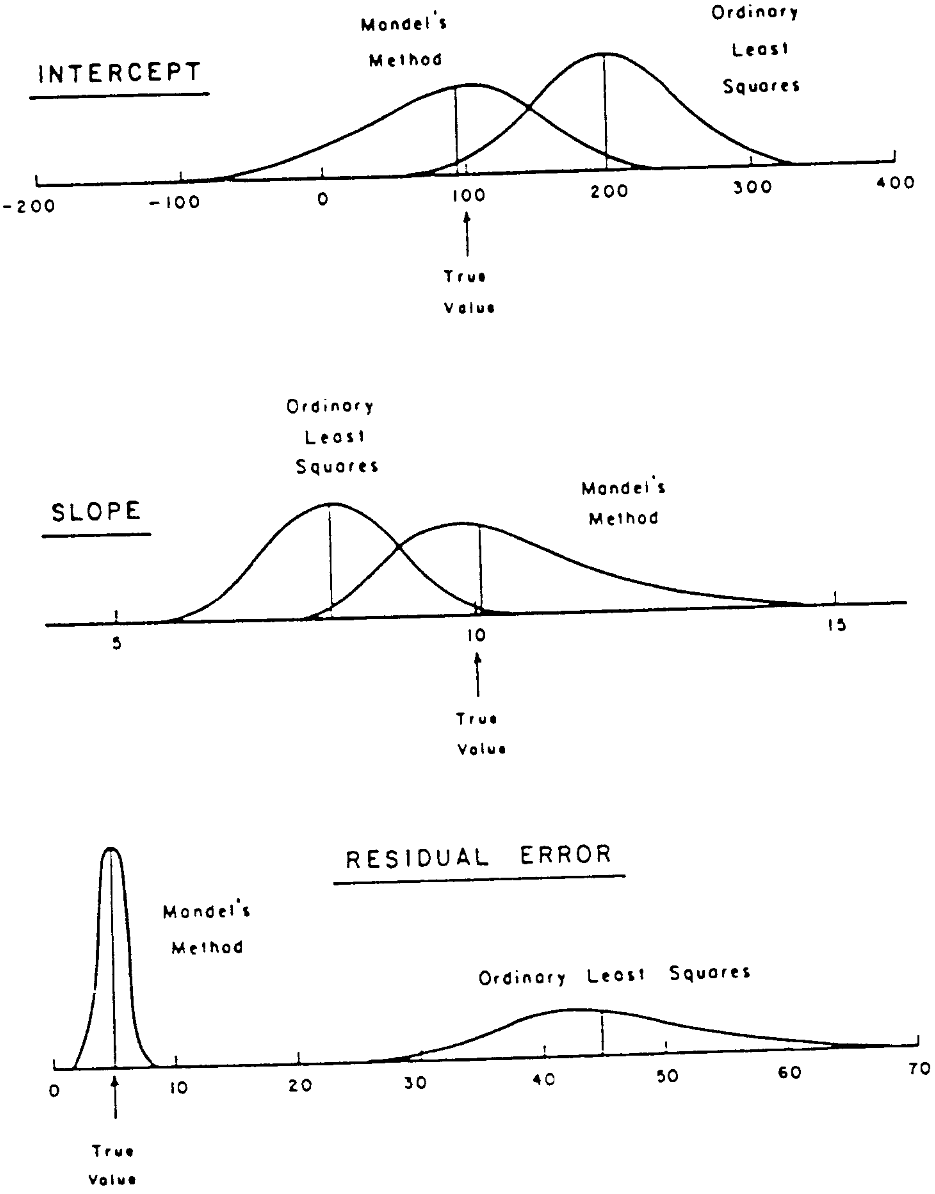


Figure 4. Comparison of distributions of regression estimates.

distributions for the intercept and the slope obtained with Mandel's method are only slightly more dispersed that those obtained by ordinary least squares. This indicates that a substantial gain in accuracy has been achieved with only a slight loss of precision. For the residual error, Mandel's method is both more accurate and more precise.

An interesting feature of Mandel's method is that, unlike the least squares technique, the same regression line will be obtained regardless of which variable is considered to be independent (X) and which is dependent (Y). Furthermore, the degree of uncertainty associated with predictions made by this method is the same for either choice of variables (2, p. 9). This property conveniently avoids a controversial aspect of the calibration application, the need to work backward through the regression procedure to estimate what value of X gave rise to an observed value of Y.

Another series of computer simulation tests was performed using the standard procedures for computing interval estimates for the intercept, slope, and σ_{yx}. A level of confidence of $1 - \alpha = 0.95$ was selected and the number of times the interval estimate actually contained the true population parameter was counted. For 1000 replications, the empirically observed results should fall within the range of approximately $0.95 \pm 2(((0.95)(0.05))/1000)^{1/2} \simeq 0.95 \pm 0.014$ when the interval estimation process is working properly. It can be seen from the results in Table 4 that, even for small amounts of X error, the interval estimates computed by ordinary least squares contain the population parameters substantially less often than desired. In contrast, all of the interval estimates computed by Mandel's method are satisfactory even though the standard confidence interval estimation procedures are known to be approximate.

Table 4. Effect of X error on interval estimates.

PARAMETER	DESIRED CONFIDENCE LEVEL	METHOD	EMPIRICALLY OBSERVED (a) CONFIDENCE LEVELS AT SELECTED LEVELS OF X ERROR (b)				
			0	25	50	75	100
Intercept (β_0=100)	0.95	Mandel	0.950	0.938	0.954	0.939	0.953
		OLS (c)	0.951	0.898	0.768	0.548	0.321
Slope (β_1=10)	0.95	Mandel	0.948	0.941	0.956	0.948	0.952
		OLS (c)	0.946	0.889	0.755	0.534	0.301
Residual Error (σ_{YX}=5)	0.95	Mandel	0.955	0.956	0.963	0.947	0.959
		OLS (c)	0.955	0.0	0.0	0.0	0.0

(a) Results obtained by computer simulation with 1000 replications of 30 data points spanning the range between approximately X = 30 and X = 70.

(b) X error is measured as the ratio σ_{XX}/σ_{YX}, expressed as a percent, in which σ_{XX} and σ_{YX} represent the precision with which the X and Y variables are measured.

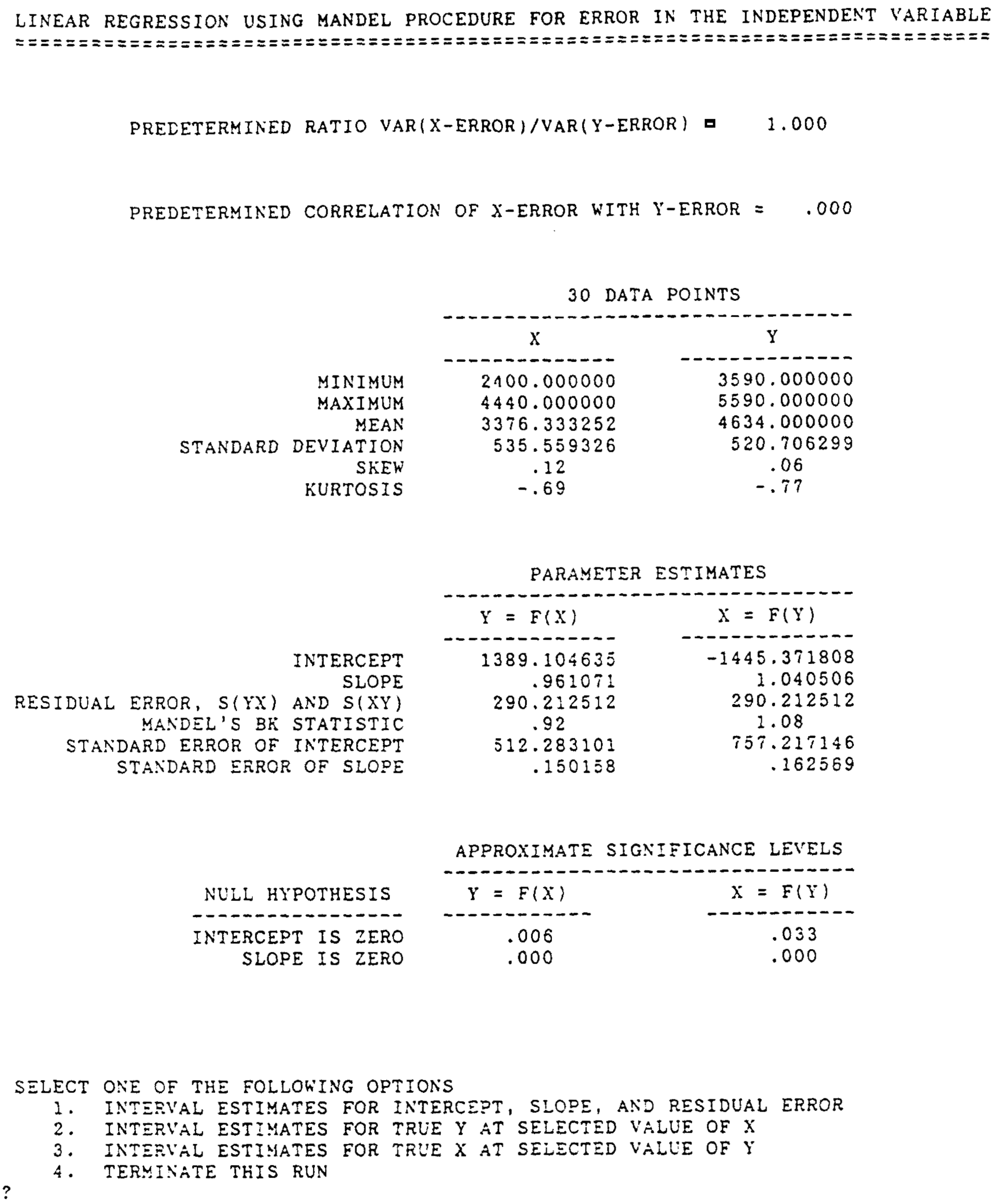
(c) Ordinary least squares.

5 USE OF INTERACTIVE SOFTWARE PACKAGE

The following example illustrates the use of the Fortran program developed to make Mandel's solution widely available to a broad range of potential users. (A complete listing of the Fortran program is contained in Appendix B.) It is based on concrete strength data collected from a construction project in New Jersey and is a contrived example in that the data set that is used was randomly generated from a population having the same statistical parameters as those observed in the field. This approach provides a known control against which the results obtained by the two methods may be compared. Otherwise, it could only be observed that the results obtained by the two methods were distinctly different and it would not be known how close either one came to estimating the true population parameters.

In order to use this program, two predetermined values must be entered. These are (a) the ratio of the X and Y error variances and (b) the degree of correlation between the X and Y errors. Note that this latter value represents the correlation between the X and Y errors, not the correlation between the X and Y measurements. Except for certain specialized applications (2, p.3), this correlation is normally zero. The ratio of error variances has been set at 1.0 because, at the relatively high levels of 7-day and 28-day strengths used in this example, it is believed that the measurement error is essentially the same for both sets of data.

Figure 5 presents the primary portion of the printout for the X-error software package. In addition to the two predetermined values, the program prints elementary statistics, basic regression parameters, and the approximate significance levels with which the intercept and slope may be



```
LINEAR REGRESSION USING MANDEL PROCEDURE FOR ERROR IN THE INDEPENDENT VARIABLE
==============================================================================


         PREDETERMINED RATIO VAR(X-ERROR)/VAR(Y-ERROR) =    1.000


         PREDETERMINED CORRELATION OF X-ERROR WITH Y-ERROR =    .000


                                                30 DATA POINTS
                                      ----------------------------------
                                             X                 Y
                                      --------------    --------------
                       MINIMUM         2400.000000       3590.000000
                       MAXIMUM         4440.000000       5590.000000
                          MEAN         3376.333252       4634.000000
            STANDARD DEVIATION          535.559326        520.706299
                          SKEW             .12               .06
                      KURTOSIS            -.69              -.77


                                             PARAMETER ESTIMATES
                                      ----------------------------------
                                          Y = F(X)          X = F(Y)
                                      --------------    --------------
                     INTERCEPT         1389.104635      -1445.371808
                         SLOPE             .961071          1.040506
RESIDUAL ERROR, S(YX) AND S(XY)         290.212512        290.212512
          MANDEL'S BK STATISTIC            .92              1.08
        STANDARD ERROR OF INTERCEPT     512.283101        757.217146
            STANDARD ERROR OF SLOPE        .150158           .162569


                                       APPROXIMATE SIGNIFICANCE LEVELS
                                      ----------------------------------
              NULL HYPOTHESIS            Y = F(X)            X = F(Y)
              -----------------       ------------        ------------
              INTERCEPT IS ZERO           .006                .033
                  SLOPE IS ZERO           .000                .000



SELECT ONE OF THE FOLLOWING OPTIONS
   1.  INTERVAL ESTIMATES FOR INTERCEPT, SLOPE, AND RESIDUAL ERROR
   2.  INTERVAL ESTIMATES FOR TRUE Y AT SELECTED VALUE OF X
   3.  INTERVAL ESTIMATES FOR TRUE X AT SELECTED VALUE OF Y
   4.  TERMINATE THIS RUN
?
```

Figure 5. Primary printout of X-error software package.

inferred to be nonzero. The data is treated as both Y = F(X) and X = F(Y) in the primary printout and also in the optional portions that follow.

As noted in the preceding section, Mandel's procedure produces the same regression line regardless of which variable is to be predicted. This can readily be demonstrated from this printout, for example, by solving for X in the equation Y = F(X) and comparing with the equation X = F(Y) obtained from the same printout.

The data points and regression results are shown in Figure 6. The estimated slope and intercept are typical in that they are examples of central values of the distributions shown in Figure 4. Like the dimensionless examples in Table 4, ordinary least squares has produced a considerably biased estimate while Mandel's method has produced an estimate very close to the true location of the line.

Figure 5 also shows the menu that is printed following the primary output. This enables the user to compute various interval estimates or, if these are not desired, to terminate the run.

If Item 1 is selected from the menu, the output shown in Figure 7 is obtained. This provides both approximate single-sided and approximate double-sided interval estimates at several commonly used significance levels for the three basic parameters of linear regression -- intercept, slope, and residual error.

Figure 8 shows the output resulting from the selection of Item 2 from the menu. In this case, the user is first required to enter the value of X at which interval estimates for Y are to be computed. The program then prints out a series of approximate single-sided and double-sided limits, exactly as was done for Item 1. The procedure for Item 3 is the same

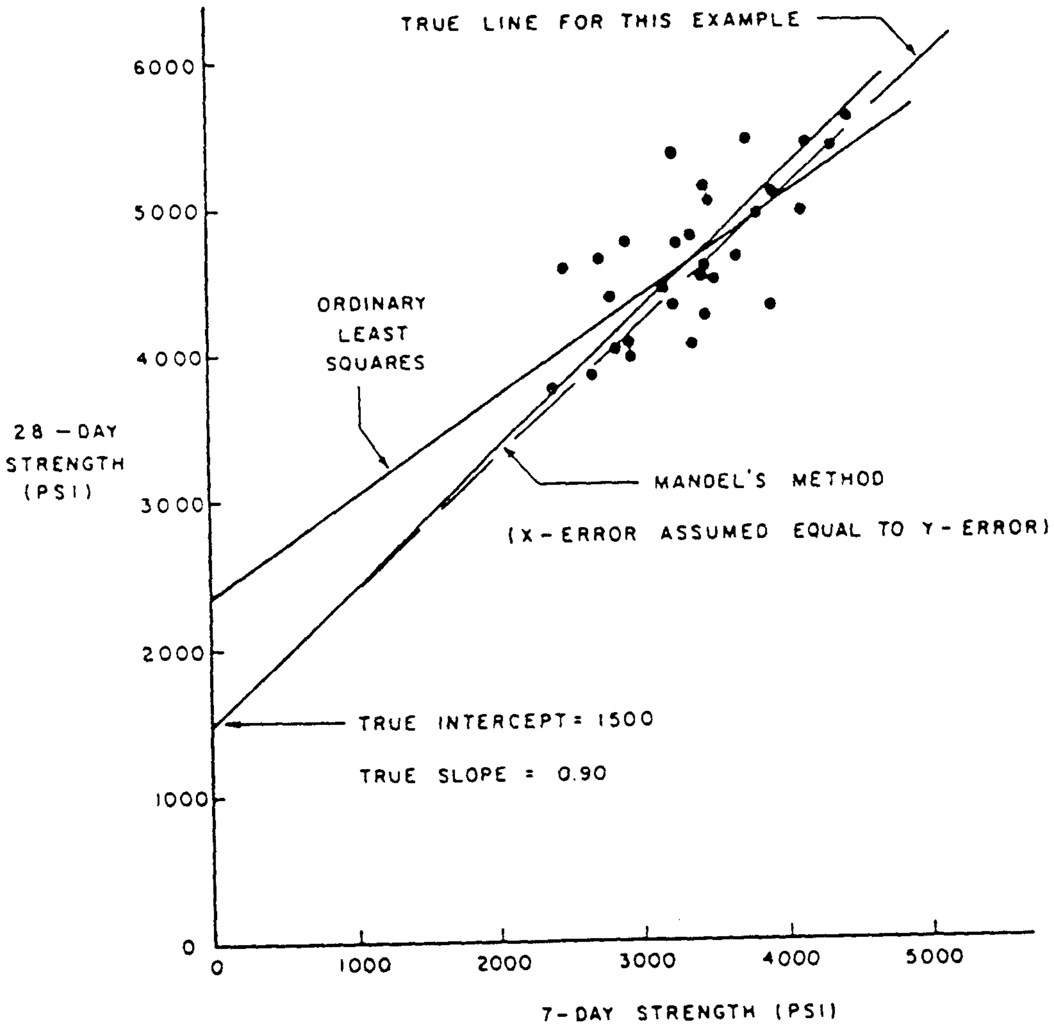


Figure 6. Typical regression results with concrete strength data.

```
      APPROXIMATE INTERVAL ESTIMATES FOR INTERCEPT
-------------------------------------------------
 TAIL AREA          Y = F(X)           X = F(Y)
-----------      --------------     --------------
LOWER  .005          -26.553672       -3537.888114
LOWER  .010          125.189853       -3313.592605
LOWER  .025          339.721983       -2996.487849
LOWER  .050          517.637687       -2733.506656
LOWER  .100          716.716942       -2439.243143
UPPER  .100         2061.492327        -451.500473
UPPER  .050         2260.571582        -157.236960
UPPER  .025         2438.487286         105.744233
UPPER  .010         2653.019416         422.848989
UPPER  .005         2804.762941         647.144498


        APPROXIMATE INTERVAL ESTIMATES FOR SLOPE
-------------------------------------------------
 TAIL AREA          Y = F(X)           X = F(Y)
-----------      --------------     --------------
LOWER  .005             .546119            .591257
LOWER  .010             .590597            .639412
LOWER  .025             .653480            .707492
LOWER  .050             .705630            .763952
LOWER  .100             .763983            .827129
UPPER  .100            1.158158           1.253883
UPPER  .050            1.216512           1.317060
UPPER  .025            1.268662           1.373520
UPPER  .010            1.331544           1.441600
UPPER  .005            1.376023           1.489755


APPROXIMATE INTERVAL ESTIMATES FOR RESIDUAL ERROR
-------------------------------------------------
 TAIL AREA           S(YX)               S(XY)
-----------      --------------     --------------
LOWER  .005          215.050256         215.050256
LOWER  .010          221.014407         221.014407
LOWER  .025          230.306780         230.306780
LOWER  .050          238.849756         238.849756
LOWER  .100          249.393031         249.393031
UPPER  .100          352.869141         352.869300
UPPER  .050          373.245117         373.245152
UPPER  .025          392.498291         392.498435
UPPER  .010          416.953857         416.953873
UPPER  .005          435.017822         435.017952


SELECT ONE OF THE FOLLOWING OPTIONS
   1.  INTERVAL ESTIMATES FOR INTERCEPT, SLOPE, AND RESIDUAL ERROR
   2.  INTERVAL ESTIMATES FOR TRUE Y AT A SELECTED VALUE OF X
   3.  INTERVAL ESTIMATES FOR TRUE X AT A SELECTED VALUE OF Y
   4.  TERMINATE THIS RUN

?
```

Figure 7. Printout resulting from Menu Item 1.

```
ENTER VALUE OF X AT WHICH COMPUTATIONS ARE MADE
?
3000


                    POINT ESTIMATE OF TRUE Y =     4272.317024


                    S(Y FOR OBSERVED X)      =      293.917236

                          APPROXIMATE INTERVAL ESTIMATES
                         --------------------------------
                          TAIL AREA       Y FOR OBSERVED XDEL
                         -----------      ----------------
                         LOWER  .005            3460.105551
                         LOWER  .010            3547.162096
                         LOWER  .025            3670.250253
                         LOWER  .050            3772.321978
                         LOWER  .100            3886.541402
                         UPPER  .100            4658.092645
                         UPPER  .050            4772.311789
                         UPPER  .025            4874.389681
                         UPPER  .010            4997.475315
                         UPPER  .005            5084.536625


SELECT ONE OF THE FOLLOWING OPTIONS
   1.  INTERVAL ESTIMATES FOR INTERCEPT, SLOPE, AND RESIDUAL ERROR
   2.  INTERVAL ESTIMATES FOR TRUE Y AT SELECTED VALUE OF X
   3.  INTERVAL ESTIMATES FOR TRUE X AT SELECTED VALUE OF Y
   4.  TERMINATE THIS RUN
?
```

Figure 8. Printout resulting from Menu Item 2.

except that, in this case, a value of Y must be entered in order to obtain approximate interval estimates for X.

The entire menu appears again after each selection is acted upon. In this way, the user may repeat any selection or, by selecting Item 4, the interactive session may be terminated.

6 NEED FOR FUTURE RESEARCH

This report has dealt entirely with linear regression applications because theoretical solutions for the more complex models apparently have not been derived or published to date. Obvious extensions of this methodology that would be immediately useful are the following:

1. Regression line forced to go through the origin.
2. Multiple linear regression and other curvilinear models.
3. Nonlinear regression models.

It was shown that results obtained by linear regression can be greatly in error when X error is present. It is reasonable to believe that similar problems exist with the more complex models. Since many applications require the line to go through the origin, or take some type of curvilinear form, it would seem highly desirable to develop these more advanced techniques in a manner that can properly account for the presence of X error.

Ideally, if the appropriate talents are brought to bear upon this problem, it is believed that a theoretical solution can be found as it was for the linear case. If for some reason a theoretical solution is not obtainable, it may still be possible to empirically derive a practical solution that can be shown to substantially improve upon the estimates that are made.

7 SUMMARY AND CONCLUSIONS

Regression analysis is frequently used in the engineering profession to develop mathematical models for many different applications. Regression theory assumes that the X values are known without error, a requirement that often cannot be met. Computer simulation was used to demonstrate that, under certain conditions, regression estimates obtained by ordinary least squares can be seriously in error. A quantitative measure of the severity of the problem is discussed in Appendix A.

The consequences of this finding may be alarming. In the presence of X error, OLS estimates may often be unreliable. The degree to which this ultimately affects the conclusions of research studies or influences policy decisions is not known, but the potential harm of specifying the wrong material or product, or of establishing a less-than-optimal policy or design, is recognized to be substantial. An error of this type will seldom be an isolated case; it will be repeated with each subsequent application of the product or design standard.

It was also demonstrated by computer simulation that Mandel's method is extremely effective at removing most of the bias introduced by error in the X variable. Figure 4 and Tables 3 and 4 clearly show that, in general, Mandel's method provides substantially more accurate results than ordinary least squares and Figure 6 illustrates this fact with a specific example based on concrete strength tests. The complete theoretical development is contained in the source document (2).

To make this technique widely available to a broad range of users, Appendix B provides the Fortran coding for an extremely user-friendly interactive program. Unlike many commercial software packages, the needs of the typical user have been anticipated to the extent that a wide variety of statistical estimates may be obtained directly from the printout without the need for external tables or specialized expertise.

To develop the full range of this important new technique, additional research will be required. Desirable extensions of this work include regression lines forced to go through the origin, various curvilinear forms, multiple linear regression, and X-error procedures coupled with nonlinear regression techniques.

8 REFERENCES

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2. J. Mandel. Fitting Straight Lines when Both Variables are Subject to Error. Journal of Quality Technology, Volume 16, No. 1, pp. 1-14, 1984.

Appendix A: SENSITIVITY ANALYSES

Computer simulation analyses were performed to determine how robust the Mandel regression procedure was with respect to: (1) mis-specified lambda (i.e. the ratio of Y error to X error variances), (2) slope variations, and (3) various ranges of X and Y observations. In this simulation the true equation subject to estimation was known, allowing the effectiveness of the Mandel procedure to be gaged by monitoring the ratios of the mean parameter estimates to their true values. A ratio of 1.0 for the slope parameter would indicate that the average estimated slope was correctly estimated as equal to the true value. A ratio of 0.5, on the other hand, would indicate that the average estimated slope was biased downwards by 50 per cent.

This sensitivity analysis required that the Mandel procedure be tested throughout the range of possible regression applications. This meant more than simply testing small and large numbers. Linear scaling of the X and Y observations may superficially change the apparent slope of the reported regression line but, in fact, linear scaling has no effect on the actual regression analysis itself other than cosmetic representation. Consequently a more meaningful criteria was developed to test the Mandel procedure.

The key to this testing procedure lay in the development of two ratios: that of the range of X observations to the residual error in X and that of the range of Y observations to the residual error in Y. The significance of these ratios may be readily appreciated by the schematic diagram presented in Figure A1. Consider the hypothetical case in which X error is three times as large as the error in Y. An excellent estimate of

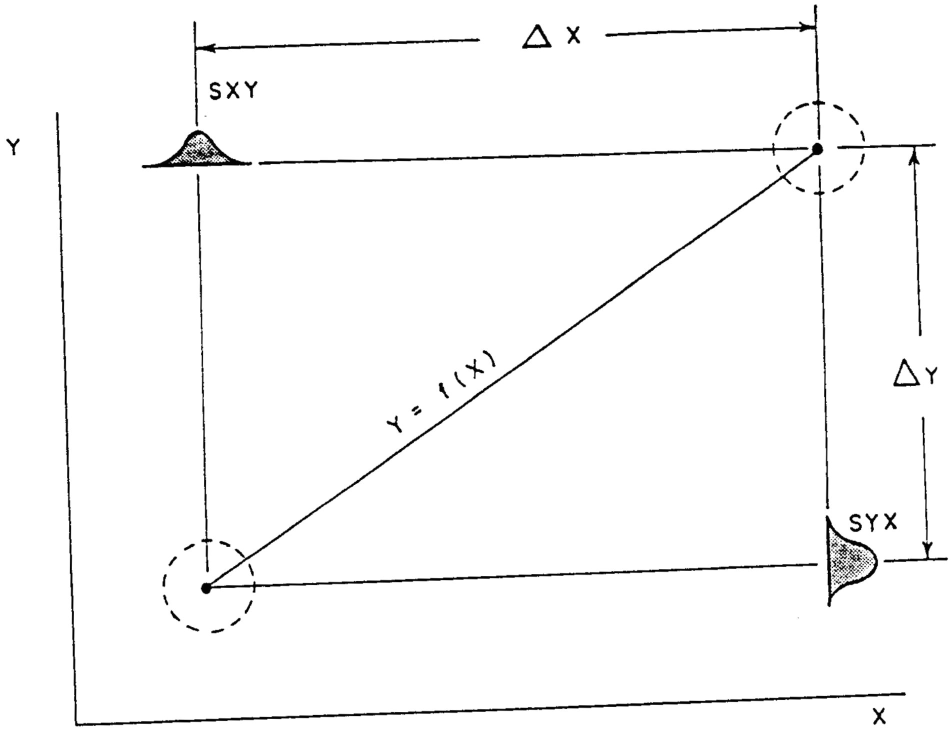

Figure A.1. Schematic representation of the influrence the ratios (ΔX/SXY) and (ΔY/SYX) have on the estimation of Y=f(X).

the Y=f(X) relationship may be obtained, even with OLS and regardless of the true slope, provided the observed data spans a range substantially broader than the potential error in the X values. That is, provided the locus of (X,Y) coordinates at one extreme end of the Y=f(X) relationship is substantially distinct from the locus of (X,Y) coordinates at the other extreme end, the line passing through the center of the two data masses will be easily identifiable and more precisely estimated as the two loci are separated.

The ratios (DELTA X)/(SXY) and (DELTA Y)/(SYX) describe the degree to which these loci are offset from one another. Ratios of 100 and 100 would be associated with relationships so strong that they may be well estimated graphically. Ratios of 5 and 5 (or lower) would be associated with visually more obscure trends. Of course, ratios of 5 and 100 or any other magnitude and combination may actually occur and can as easily be checked.

The equation first presented in Table 1 of this report, Y = 100.0 + 10.0*X, was used as the nominal point of departure for this sensitivity analysis. Lambda was first correctly specified, then over and under estimated by a factor of two, and then over estimated by a factor of ten thousand (essentially ignoring X error and approximating an OLS analysis). One thousand data sets each for the sample sizes of 10, 30 and 100 (X,Y) coordinate pairs were generated under a variety of (DELTA X)/(SXY) and (DELTA Y)/(SYX) conditions. The Mandel regression procedure was applied to each data set, and the accuracy of the parameter estimates was monitored as previously described. Additionally, a tally was kept to empirically determine the frequency with which the true parameter values fell outside their estimated 95 percent confidence limits. Then, for even more extreme conditions, the entire sensitivity analysis was reproduced with the slope both increased and decreased fifty-fold, using the true relationships of $Y = 100 + 500.0*X$ and $Y = 100 + 0.2*X$, respectively. Partial results from these investigations are presented in Tables A1 and A2.

Several general conclusions may be drawn from this sensitivity analysis. They are:

1) Properly used, the Mandel procedure is accurate and generally reliable regardless of the data's variability.
2) Proper use of the Mandel procedure produces significance levels quite close to the nominal level specified. Apparently the Student t distribution used for this approximation is very close to the as-yet-unknown statistical distribution appropriate for this step.
3) The intercept appears to be the most sensitive of all estimated parameters, especially when the origin lies substantially outside the data range. This is intuitively an expected result, however, as in such cases the intercept is an extrapolated value. Thus this sensitivity is not considered to represent a significant shortcoming.
4) Mis-specification of lambda has relatively little effect in degrading the accuracy of slope parameter estimates, especially when the data spread is large relative to the inherent error. The intercept estimates are somewhat more affected, especially when the origin lies well outside the range of the data.
5) Mis-specification of lambda necessarily biases at least one of the error estimates - - either SYX or SXY, or both. This is mathematically unavoidable.

Table A1. Sensitivity of Mandel's mean parameter estimates.

				(MEAN PARAMETER ESTIMATE) / (TRUE PARAMETER VALUE)											
				Y = 100.0 + 10.0*X, SYX=5.0				Y = 100.0 + 500.0*X, SYX=1.0				Y = 100.0 + 0.2*X, SYX=1.0			
				TRUE/ESTIMATED LAMBDA				TRUE/ESTIMATED LAMBDA				TRUE/ESTIMATED LAMBDA			
PARAMETER	ΔX/SX	ΔY/SY	n	1.0	2.0	0.5	0.0001	1.0	2.0	0.5	0.0001	1.0	2.0	0.5	0.0001
INTERCEPT	10	100	30	0.99	0.97	1.00	1.47	0.52	1.32	0.94	1.40	1.00	1.00	1.00	1.00
	10	100	100	0.99	0.99	1.00	1.53	0.74	0.83	1.28	1.24	1.00	1.00	1.00	1.00
	5	5	30	0.95	0.49	1.34	1.91	-1.62	-28.97	19.81	46.57	1.00	0.99	1.01	0.03
	5	5	100	0.99	0.46	1.42	1.98	-0.44	-30.08	24.34	54.37	1.00	0.99	1.01	1.03
SLOPE	10	100	30	1.00	1.01	1.00	0.91	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	10	100	100	1.00	1.00	1.00	0.90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	5	5	30	1.02	1.16	0.89	0.71	1.02	1.20	0.87	0.70	1.02	1.16	0.90	0.70
	5	5	100	1.00	1.17	0.86	0.68	1.01	1.21	0.84	0.64	1.00	1.17	0.87	0.69
SYX^2	10	100	30	1.00	0.50	1.98	84.13	1.01	0.98	1.01	1.01	1.01	0.98	1.97	1.01
	10	100	100	1.00	0.50	1.97	84.07	1.01	0.99	1.01	1.01	1.00	0.99	1.99	1.01
	5	5	30	0.99	0.66	1.28	1.70	0.98	0.65	1.30	1.69	1.00	0.65	1.30	1.71
	5	5	100	1.00	0.65	1.30	1.68	1.00	0.64	1.29	1.64	1.00	0.65	1.30	1.68
SXY^2	10	100	30	1.00	1.00	0.99	0.08	1.01	1.96	0.50	0.00	1.01	1.97	0.99	0.00
	10	100	100	1.00	1.01	0.99	0.08	1.01	1.97	0.50	0.00	1.00	1.98	1.00	0.00
	5	5	30	0.99	1.32	0.64	0.00	0.98	1.30	0.65	0.00	1.00	1.29	0.65	0.00
	5	5	100	1.00	1.29	0.65	0.00	1.00	1:28	0.64	0.00	1.00	1.31	0.65	0.00

Notes:

1) Computer simulation used to perform sensitivity analysis. Values in body of table represent the average of 1000 replicate runs. Values of 1.0 indicate no apparent bias.

2) LAMBDA = $(SYX/SXY)^2$

3) ΔX/SX = (Range of true X values)/SXY

4) ΔY/SY = (Range of true Y values)/SYX

Table A2. Empirical significance levels for Mandel's parameter estimates.

				EMPIRICAL SIGNIFICANCE LEVELS, 0.05 ERROR RATE DESIRED											
				Y = 100.0 + 10.0*X, SYX=5.0				Y = 100.0 + 500.0*X, SYX=1.0				Y = 100.0 + 0.2*X, SYX=1.0			
				TRUE/ESTIMATED LAMBDA				TRUE/ESTIMATED LAMBDA				TRUE/ESTIMATED LAMBDA			
PARAMETER	ΔX/SX	ΔY/SY	n	1.0	2.0	0.5	0.0001	1.0	2.0	0.5	0.0001	1.0	2.0	0.5	0.0001
INTERCEPT	10	100	30	0.04	0.06	0.06	0.31	0.04	0.05	0.04	0.04	0.04	0.07	0.05	0.04
	10	100	100	0.05	0.05	0.05	0.84	0.04	0.06	0.05	0.05	0.05	0.06	0.05	0.05
	5	5	30	0.09	0.07	0.20	0.55	0.10	0.07	0.22	0.57	0.11	0.07	0.20	0.58
	5	5	100	0.11	0.30	0.45	0.99	0.12	0.36	0.47	0.99	0.09	0.30	0.42	0.99
SLOPE	10	100	30	0.04	0.06	0.05	0.34	0.04	0.05	0.04	0.04	0.04	0.07	0.06	0.04
	10	100	100	0.05	0.05	0.04	0.87	0.04	0.06	0.05	0.05	0.05	0.06	0.06	0.04
	5	5	30	0.09	0.07	0.20	0.56	0.10	0.07	0.22	0.57	0.11	0.08	0.19	0.59
	5	5	100	0.10	0.30	0.46	0.99	0.12	0.36	0.47	0.99	0.09	0.32	0.44	0.99
SYX^2	10	100	30	0.05	0.66	0.74	1.00	0.05	0.05	0.04	0.05	0.05	0.04	0.76	0.05
	10	100	100	0.05	1.00	1.00	1.00	0.05	0.05	0.06	0.05	0.04	0.04	1.00	0.06
	5	5	30	0.05	0.28	0.19	0.57	0.06	0.29	0.19	0.55	0.04	0.31	0.18	0.57
	5	5	100	0.05	0.86	0.49	0.96	0.05	0.85	0.44	0.95	0.03	0.82	0.49	0.96
SXY^2	10	100	30	0.05	0.05	0.06	1.00	0.05	0.76	0.66	1.00	0.05	0.75	0.05	1.00
	10	100	100	0.05	0.06	0.05	1.00	0.05	1.00	1.00	1.00	0.04	1.00	0.05	1.00
	5	5	30	0.05	0.21	0.35	1.00	0.06	0.20	0.31	1.00	0.04	0.21	0.29	1.00
	5	5	100	0.05	0.48	0.83	1.00	0.05	0.44	0.86	1.00	0.03	0.49	0.82	1.00

Notes:

1) Computer simulation used to perform sensitivity analysis. Values in body of table represent the error proportion of 1000 replicate runs. Values of 0.05 indicate nominal risk level achieved.

2) LAMBDA = $(SYX/SXY)^2$

3) ΔX/SX = (Range of true X values)/SXY

4) ΔY/SY = (Range of true Y values)/SYX

6) Parameter estimates using OLS when the Mandel procedure is appropriate (i.e., extreme mis-specification of lambda,) may be generally tolerable provided the data spread is large relative to the inherent variablility.

7) Inferences about estimated parameters become progressively less reliable as the degree to which lambda is mis-specified increases. In the most extreme case, where the procedure essentially defaults to OLS, not only are the nominal risk levels understated but they become progressively worse as the sample size is increased.

In summary, these sensitivity tests tend to confirm that, properly used, the Mandel regression procedure is and accurate, precise, and powerful tool over a very broad range of potential applications. While it can hardly be recommended that any tool be misused, it also appears that this particular regression procedure is moderately forgiving of inadvertant abuse. Thus it is highly recommended for immediate implementation wherever appropriate.

APPENDIX B: Fortran Coding For Mandel Subroutine

```
C=======================================================================
C
C                  SUBROUTINE MANDEL(N,X,Y,VXY,RHO,PRINT)
C
C
C
C   ACKNOWLEDGEMENT
C   ---------------
C
C          THEORETICAL DEVELOPMENT -
C                  DR. JOHN MANDEL, NATIONAL BUREAU OF STANDARDS
C
C          FORTRAN CODING -
C                  RICARDO BARROS, WITH THE ASSISTANCE OF RICHARD WEED,
C                  BOTH OF THE NEW JERSEY DEPT. OF TRANSPORTATION.
C                  AUGUST 22, 1988.
C
C
C   SUMMARY
C   -------
C
C          THIS SUBROUTINE ESTIMATES AND, OPTIONALLY, REPORTS THE
C          RELEVANT PARAMETERS OF THE REGRESION LINES Y=F(X) AND X=F(Y)
C          WHEN BOTH X AND Y VARIABLES ARE SUBJECT TO MEASUREMENT ERROR.
C
C          AS THE X ERROR APPROACHES ZERO, THE ESTIMATED INTERCEPT AND
C          SLOPE PARAMETERS PRODUCED BY MANDEL APPROACH THE VALUES
C          PRODUCED BY CONVENTIONAL ORDINARY LEAST SQUARES TECHNIQUES.
C
C          THE SOURCE DOCUMENT FOR THE UNDERLYING THEORY IS 'FITTING
C          STRAIGHT LINES WHEN BOTH VARIABLES ARE SUBJECT TO ERROR' BY
C          DR. JOHN MANDEL AND PUBLISHED IN THE JOURNAL OF QUALITY
C          TECHNOLOGY, VOL. 16, NO. 1, JANUARY 1984.
C
C          FURTHER DISCUSSION OF THE X ERROR PROBLEM IS PRESENTED IN
C          'APPLIED REGRESSION IN THE PRESENCE OF X-ERROR'
C          BY RICARDO BARROS AND RICHARD WEED OF THE NEW JERSEY
C          DEPARTMENT OF TRANSPORTATION.
C
C          THIS SUBROUTINE REQUIRES LESS THAN 0.2 SECONDS TO ESTIMATE
C          REGRESSION PARAMETERS FOR AN ARRAY OF 1000 (X,Y) DATA
C          OBSERVATIONS ON AN IBM 3081.
C
C
C   VARIABLE DICTIONARY
C   -------------------
C
C
C          CALLING ARGUMENTS:
C          ------------------
C
C
C          N       NUMBER OF (X,Y) COORDINATES IN SAMPLE.
C                  NOTE: MAXIMUM N=1000.
C
```

```
C           X        THE ONE-DIMENSIONAL ARRAY OF X OBSERVATIONS.
C
C           Y        THE ONE-DIMENSIONAL ARRAY OF Y OBSERVATIONS.
C
C           VXY      THE RATIO OF THE PRECISION VARIANCES IN MEASURED
C                    VALUES  OF X AND Y.  VXY=VAR(X)/VAR(Y).
C
C           RHO      CORRELATION OF X ERROR WITH Y ERROR.
C                    0.0 <= RHO <= 1.0.  RHO= 0.0 WHEN X AND Y ERRORS
C                    ARE INDEPENDENT.
C
C           PRINT    AN INDICATOR VARIABLE DETERMINING WHETHER
C                    ESTIMATED PARAMETERS ARE TO BE REPORTED AT USER'S
C                    TERMINAL.  SETTING PRINT=0 SUPPRESSES OUTPUT, PRINT=1
C                    DISPLAYS ESTIMATES.  IN EITHER CASE, ALL PARAMETER
C                    ESTIMATES ARE AVAILABLE IN COMMON SPACE.
C
C
C           INTERNAL VARIABLES AVAILABLE WITHIN SUBROUTINE.
C           ----------------------------------------------
C
C                                 Y=F(X)
C                                 ------
C
C           ALPHAT   ('ALPHA HAT') ESTIMATED INTERCEPT OF REGRESSION
C                     LINE.
C
C           BETHAT   ('BETA HAT') ESTIMATED SLOPE OF REGRESSION LINE.
C
C           STDALP   ('STD. DEV. SUB ALPHA HAT') STANDARD ERROR OF
C                    INTERCEPT ESTIMATE.
C
C           STDBET   ('STD. DEV. SUB BETA HAT') STANDARD ERROR
C                    OF SLOPE ESTIMATE.
C
C           STDDLT   ('STD. DEV. SUB DELTA') ESTIMATED PRECISION
C                    OF X MEASUREMENTS.
C
C           STDEPS   ('STD. DEV. SUB EPSILON') ESTIMATED PRECISION OF
C                    Y MEASUREMENTS. THE RESIDUAL ERROR IN CONVENTIONAL
C                    ORDINARY LEAST SQARES PROCEDURE.
C
C           B        SLOPE ESTIMATE.
C
C           K       TANGENT OF ANGLE ALONG WHICH RESIDUALS MINIMIZED.
C
C           BK1      MANDEL'S STATISTIC GAUGING NECESSITY FOR SUBROUTINE
C                    MANDEL IN PLACE OF ORDINARY LEAST SQUARES PROCEDURE.
C
C           XO       A FUTURE MEASURED X VALUE.
C
C           YO       THE PREDICTED Y VALUE GIVEN XO.
C
C           STDVYO   STANDARD ERROR OF PREDICTED YO.
C
```

```
C         STDE    ('STDV. DEV. SUB E') STANDARD ERROR OF ESTIMATE ON    MANO1110
C                 TRANSFORMED SCALE.                                    MANO1120
C                                                                       MANO1130
C         SSUU    SUMS OF SQUARES, TRANSFORMED VARIABLE U.              MANO1140
C                                                                       MANO1150
C         SSVV    SUMS OF SQUARES, TRANSFORMED VARIABLE V.              MANO1160
C                                                                       MANO1170
C         C1-C5   CONVENIENT INTERMEDIATE VARIABLES USED TO CALCULATE   MANO1180
C                 STANDARD ERROR OF ESTIMATE FOR FUTURE VALUES.         MANO1190
C                                                                       MANO1200
C                               X=F(Y)                                  MANO1210
C                               ------                                  MANO1220
C                                                                       MANO1230
C         ALPH2   ('ALPHA HAT 2') ESTIMATED INTERCEPT OF  REGRESSION    MANO1240
C                 LINE.                                                 MANO1250
C                                                                       MANO1260
C         BETH2   ('BETA HAT 2') ESTIMATED SLOPE OF  REGRESSION LINE.   MANO1270
C                                                                       MANO1280
C         STDAL2  ('STD. DEV. SUB ALPHA 2') STANDARD ERROR OF           MANO1290
C                 INTERCEPT ESTIMATE.                                   MANO1300
C                                                                       MANO1310
C         STDB2   ('STD. DEV. SUB BETA 2') STANDARD ERROR OF            MANO1320
C                 SLOPE ESTIMATE.                                       MANO1330
C                                                                       MANO1340
C         STDD2   ('STD. DEV. SUB DELTA 2') ESTIMATED PRECISION OF      MANO1350
C                 THE Y MEASUREMENTS.                                   MANO1360
C                                                                       MANO1370
C         STDEP2  ('STD. DEV. SUB EPSILON 2') ESTIMATED PRECISION OF    MANO1380
C                 THE X MEASUREMENTS.                                   MANO1390
C                                                                       MANO1400
C         B2      SLOPE ESTIMATE.                                       MANO1410
C                                                                       MANO1420
C         K2     TANGENT OF ANGLE ALONG WHICH RESIDUALS MINIMIZED.      MANO1430
C                                                                       MANO1440
C         BK2     MANDEL'S STATISTIC GAUGING NECESSITY FOR SUBROUTINE   MANO1450
C                 MANDEL IN PLACE OF ORDINARY LEAST SQUARES PROCEDURE.  MANO1460
C                                                                       MANO1470
C         YO      A FUTURE MEASURED Y VALUE.                            MANO1480
C                                                                       MANO1490
C         XO      THE PREDICTED X VALUE GIVEN YO.                       MANO1500
C                                                                       MANO1510
C         STDVXO  STANDARD ERROR OF PREDICTED XO.                       MANO1520
C                                                                       MANO1530
C         STDE2   ('STD. DEV. SUB E 2') STANDARD ERROR OF ESTIMATE ON   MANO1540
C                 TRANSFORMED SCALE.                                    MANO1550
C                                                                       MANO1560
C         SSUU2   SUMS OF SQUARES 2, TRANSFORMED VARIABLE U.            MANO1570
C                                                                       MANO1580
C         SSVV2   SUMS OF SQUARES 2, TRANSFORMED VARIABLE V.            MANO1590
C                                                                       MANO1600
C         C6-C10  CONVENIENT INTERMEDIATE VARIABLES USED TO CALCULATE   MANO1610
C                 STANDARD ERROR OF ESTIMATE FOR FUTURE VALUES.         MANO1620
C                                                                       MANO1630
C                                                                       MANO1640
C                        BOTH Y=F(X) AND X=F(Y)                         MANO1650
```

```
C                        ---------------------                          MANO1660
C                                                                       MANO1670
C          SSXX    SUMS OF SQUARES, X.                                  MANO1680
C                                                                       MANO1690
C          SSYY    SUMS OF SQUARES, Y.                                  MANO1700
C                                                                       MANO1710
C          SSXY    CROSS PRODUCT SUMS OF SQUARES, X & Y.                MANO1720
C                                                                       MANO1730
C                                                                       MANO1740
C=======================================================================MANO1750
C                                                                       MANO1760
C                                                                       MANO1770
C          AUXILLARY SUBROUTINES AND REQUIRED ARGUMENTS FOLLOW.         MANO1780
C                                                                       MANO1790
C    SUBROUTINE REPORT(N,X,Y,RHO)                                       MANO1800
C                                                                       MANO1810
C               WHERE CALLING ARGUMENTS ARE SAME AS ABOVE.  THIS        MANO1820
C               SUBROUTINE CONTAINS FORMAT STATEMENTS NECESSARY         MANO1830
C               TO PRINT OUT PARAMETER ESTIMATES AND MAKES USE OF       MANO1840
C               COMMON SPACE.                                           MANO1850
C                                                                       MANO1860
C    SUBROUTINE CONF1(N)                                                MANO1870
C                                                                       MANO1880
C               WHERE N IS THE SAMPLE SIZE.  THIS SUBROUTINE            MANO1890
C               ESTIMATES AND REPORTS THE CONFIDENCE INTERVALS FOR      MANO1900
C               THE SLOPE, INTERCEPT, AND RESIDUAL ERROR PARAMETERS AND MANO1910
C               MAKES USE OF COMMON SPACE.                              MANO1920
C                                                                       MANO1930
C    SUBROUTINE CONF2(N,INPUT)                                          MANO1940
C                                                                       MANO1950
C               WHERE N IS THE SAMPLE SIZE AND INPUT IS THE MENU        MANO1960
C               SELECTION.  THIS SUBROUTINE ESTIMATES AND REPORTS       MANO1970
C               CONFIDENCE INTERVALS FOR Y AT SELECTED VALUES OF X      MANO1980
C               AND VICE VERSA.   IT ALSO MAKES USE OF COMMON SPACE.    MANO1990
C                                                                       MANO2000
C    SUBROUTINE TTABLE(NDF,T,AREA,MODE)                                 MANO2010
C                                                                       MANO2020
C               WHERE TTABLE REFERS TO STUDENT T DISTRIBUTION AND       MANO2030
C               NDF = DEGREES OF FREEDOM,                               MANO2040
C               T = STUDENT T STATISTIC,                                MANO2050
C               AREA = AREA MEASURED FROM LEFT SIDE OF DISTRIBUTION,    MANO2060
C               MODE = 1 FOR T IN AND AREA OUT, AND                     MANO2070
C               MODE=2 FOR AREA IN AND T OUT.                           MANO2080
C                                                                       MANO2090
C                                                                       MANO2100
C    SUBROUTINE CSTABL(NDF,CHISQ,AREA,MODE)                             MANO2110
C                                                                       MANO2120
C               WHERE CSTABL REFERS TO CHI-SQUARE DISTRIBUTION AND      MANO2130
C               CHISQ = CHI-SQUARE STATISTIC,                           MANO2140
C               AREA = AREA MEASURED FROM LEFT SIDE OF DISTRIBUTION,    MANO2150
C               MODE = 1 FOR CHISQ IN AND AREA OUT, AND                 MANO2160
C               MODE = 2 FOR AREA IN AND CHISQ OUT.                     MANO2170
C                                                                       MANO2180
C                                                                       MANO2190
C    SUBROUTINE ZTABLE(Z,AREA,MODE)                                     MANO2200
```

```
C                                                                       MANO2210
C               WHERE ZTABLE REFERS TO THE NORMAL DISTRIBUTION AND      MANO2220
C               Z = THE Z-SCORE STATISTIC                               MANO2230
C               AREA = AREA MEASURED FROM LEFT SIDE OF DISTRIBUTION,    MANO2240
C               MODE = 1 FOR Z-SCORE STATISTIC IN AND AREA OUT.         MANO2250
C               MODE = 2 FOR AREA IN AND Z-SCORE STATISTIC OUT.         MANO2260
C                                                                       MANO2270
C                                                                       MANO2280
C    SUBROUTINE STAT(Y,N,AVGY,SDY,YMIN,YMAX,YSKEW,YKURT)                MANO2290
C                                                                       MANO2300
C               WHERE THE FOLOWING DESCRIPTIVE STATISTICS ARE           MANO2310
C               GENERATED FOR THE N OBSERVATIONS OF THE ARRAY Y:        MANO2320
C               AVGY = AVERAGE                                          MANO2330
C               SDY = STANDARD DEVIATION                                MANO2340
C               YMIN = MINIMUM VALUE                                    MANO2350
C               YMAX = MAXIMUM VALUE                                    MANO2360
C               YSKEW = SKEW STATISTIC                                  MANO2370
C               YKURT = KURTOSIS STATISTIC                              MANO2380
C                                                                       MANO2390
C=======================================================================MANO2400
C                                                                       MANO2410
      SUBROUTINE MANDEL(N,X,Y,VXY,RHO,PRINT)                            MANO2420
      COMMON ALPH2,ALPHAT,B,B2,BETH2,BETHAT,BK1,BK2,                    MANO2430
     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,QUAN1,                        MANO2440
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,         MANO2450
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,      MANO2460
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,        MANO2470
     *VARYX,XO,XBAR,YO,YBAR                                             MANO2480
      DOUBLE PRECISION ALPH2,ALPHAT,B,B2,BETH2,BETHAT,                  MANO2490
     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,PRINT,QUAN1,                  MANO2500
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,         MANO2510
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,      MANO2520
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,        MANO2530
     *VARYX,XO,XBAR,YO,YBAR                                             MANO2540
      REAL BK1,BK2,U(1000),V(1000),X(N),Y(N)                            MANO2550
1     CONTINUE                                                          MANO2560
      IF(VXY.LE.0.)GO TO 30                                             MANO2570
      IF(N.LT.3.OR.N.GT.1000)GO TO 40                                   MANO2580
      IF(RHO.LT.0..OR.RHO.GT.1.)GO TO 50                                MANO2590
C                                                                       MANO2600
C  INITIALIZE AND DETERMINE GENERAL STATISTICS                          MANO2610
C                                                                       MANO2620
      SUMX=0.                                                           MANO2630
      SUMY=0.                                                           MANO2640
      DO 10 I=1,N                                                       MANO2650
      SUMX=SUMX+X(I)                                                    MANO2660
10    SUMY=SUMY+Y(I)                                                    MANO2670
      XBAR=SUMX/FLOAT(N)                                                MANO2680
      YBAR=SUMY/FLOAT(N)                                                MANO2690
      VARXY=VXY                                                         MANO2700
      VARYX=1./VARXY                                                    MANO2710
      THETA=0.                                                          MANO2720
      IF(RHO.GT.0.)THETA=RHO*DSQRT(VARYX)                               MANO2730
      THETA2=0.                                                         MANO2740
      IF(RHO.GT.0.)THETA2=RHO*DSQRT(VARXY)                              MANO2750
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      SSXX=0.                                                           MANO2760
      SSYY=0.                                                           MANO2770
      SSXY=0.                                                           MANO2780
      DO 20 I=1,N                                                       MANO2790
      SSXX=(X(I)-XBAR)**2+SSXX                                          MANO2800
      SSYY=(Y(I)-YBAR)**2+SSYY                                          MANO2810
20    SSXY=(X(I)-XBAR)*(Y(I)-YBAR)+SSXY                                 MANO2820
C                                                                       MANO2830
C  CALCULATE PARAMETER ESTIMATES FOR Y=F(X) RELATIONSHIP                MANO2840
C                                                                       MANO2850
      QUAN1=(VARYX*SSXX-SSYY)**2-4.*(VARYX*SSXY-SSYY*THETA)*(THETA*SSXX-MANO2860
     *SSXY)                                                             MANO2870
      QUAN1=DSQRT(QUAN1)                                                MANO2880
      QUAN2=SSYY-VARYX*SSXX                                             MANO2890
      QUAN3=2.*(SSXY*VARYX-SSYY*THETA)                                  MANO2900
      K=(QUAN2+QUAN1)/QUAN3                                             MANO2910
      B=(SSXY+K*SSYY)/(SSXX+K*SSXY)                                     MANO2920
      SSUU=SSXX+2.*K*SSXY+K*K*SSYY                                      MANO2930
      SSVV=B*B*SSXX-2.*B*SSXY+SSYY                                      MANO2940
      STDE=DSQRT(SSVV/(FLOAT(N-2)))                                     MANO2950
      STDDLT=STDE/DSQRT(B**2+VARYX-2.*B*THETA)                          MANO2960
      STDEPS=STDDLT*DSQRT(VARYX)                                        MANO2970
      STDALP=DSQRT(1./FLOAT(N)+(XBAR*XBAR*(1.+K*B)**2)/SSUU)*STDE       MANO2980
      STDBET=(DABS(1.+K*B)/DSQRT(SSUU))*STDE                            MANO2990
      BETHAT=B                                                          MANO3000
      ALPHAT=YBAR-BETHAT*XBAR                                           MANO3010
      BK1=K*B                                                           MANO3020
C                                                                       MANO3030
C  CALCULATE PARAMETER ESTIMATES FOR X=F(Y) RELATIONSHIP                MANO3040
C  (NOTE SUFFIX '2' ATTACHED TO VARIABLE NAMES)                         MANO3050
C                                                                       MANO3060
      QUAN4=(VARXY*SSYY-SSXX)**2-4.*(VARXY*SSXY-SSXX*THETA2)*(THETA2*   MANO3070
     *SSYY-SSXY)                                                        MANO3080
      QUAN4=DSQRT(QUAN4)                                                MANO3090
      QUAN5=SSXX-VARXY*SSYY                                             MANO3100
      QUAN6=2.*(SSXY*VARXY-SSXX*THETA2)                                 MANO3110
      K2=(QUAN5+QUAN4)/QUAN6                                            MANO3120
      B2=(SSXY+K2*SSXX)/(SSYY+K2*SSXY)                                  MANO3130
      BK2=K2*B2                                                         MANO3140
      SSUU2=SSYY+2.*K2*SSXY+K2*K2*SSXX                                  MANO3150
      SSVV2=B2*B2*SSYY-2.*B2*SSXY+SSXX                                  MANO3160
      STDE2=DSQRT(SSVV2/(FLOAT(N-2)))                                   MANO3170
      STDD2=STDE2/DSQRT(B2**2+VARXY-2.*B2*THETA2)                       MANO3180
      STDEP2=STDD2*DSQRT(VARXY)                                         MANO3190
      STDAL2=DSQRT(1./FLOAT(N)+(YBAR*YBAR*(1.+K2*B2)**2)/SSUU2)*STDE2   MANO3200
      STDB2=(DABS(1.+K2*B2)/DSQRT(SSUU2))*STDE2                         MANO3210
      BETH2=B2                                                          MANO3220
      ALPH2=XBAR-BETH2*YBAR                                             MANO3230
      YO=ALPHAT+BETHAT*XO                                               MANO3240
      C1=YBAR-BETHAT*XBAR                                               MANO3250
      C2=BETHAT                                                         MANO3260
      C3=(BETHAT*STDDLT)**2+STDE*STDE/FLOAT(N)                          MANO3270
      C4=(STDE*(1.+K*B))**2/SSUU                                        MANO3280
      C5=(XO-XBAR)**2                                                   MANO3290
      STDVYO=DSQRT(C3+C4*C5)                                            MANO3300
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      C6=XBAR-BETH2*YBAR                                                MANO3310
      C7=BETH2                                                          MANO3320
      C8=(BETH2*STDD2)**2+STDE2*STDE2/FLOAT(N)                          MANO3330
      C9=(STDE2*(1.+K2*B2))**2/SSUU2                                    MANO3340
      C10=(YO-YBAR)**2                                                  MANO3350
      STDVXO=DSQRT(C8+C9+C10)                                           MANO3360
      IF(PRINT.EQ.O.)RETURN                                             MANO3370
      CALL REPORT(N,X,Y,RHO)                                            MANO3380
      RETURN                                                            MANO3390
C                                                                       MANO3400
C  DIAGNOSTIC MESSAGES                                                  MANO3410
C                                                                       MANO3420
30    CONTINUE                                                          MANO3430
      WRITE(6,300)                                                      MANO3440
300   FORMAT(//,1X,'SUBROUTINE MANDEL REQUIRES THE RATIO VAR(X-ERROR)/VAMANO3450
     *R(Y-ERROR)',/,1X,                                                 MANO3460
     *'TO BE GREATER THAN ZERO.  IF IT IS DESIRED TO USE SUBROUTINE',/, MANO3470
     *1X,'MANDEL TO APPROXIMATE AN ORDINARY LEAST SQUARES SOLUTION, ENTEMANO3480
     *R',/,1X,                                                          MANO3490
     *'A VERY SMALL POSITIVE VALUE FOR THIS RATIO.  OTHERWISE ENTER',/, MANO3500
     *1X,'0.0 TO TERMINATE THIS RUN.')                                  MANO3510
      READ(5,*)VXY                                                      MANO3520
      IF(VXY.LE.O.)STOP                                                 MANO3530
      GO TO 1                                                           MANO3540
40    CONTINUE                                                          MANO3550
      WRITE(6,400)                                                      MANO3560
400   FORMAT(//,1X,'MAXIMUM PERMISSIBLE SAMPLE SIZE N = 1000 IN ',      MANO3570
     &'SUBROUTINE MANDEL.',                                             MANO3580
     &/,1X,'MINIMUM PERMISSIBLE SAMPLE SIZE N = 3 IN',                  MANO3590
     &' SUBROUTINE MANDEL.',//)                                         MANO3600
      STOP                                                              MANO3610
50    CONTINUE                                                          MANO3620
      WRITE(6,500)                                                      MANO3630
500   FORMAT(//,1X,'ILLEGAL CALLING ARGUMENT IN SUBROUTINE MANDEL.',//) MANO3640
      STOP                                                              MANO3650
      END                                                               MANO3660
C                                                                       MANO3670
      SUBROUTINE REPORT(N,X,Y,RHO)                                      MANO3680
      COMMON ALPH2,ALPHAT,B,B2,BETH2,BETHAT,BK1,BK2,                    MANO3690
     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,QUAN1,                        MANO3700
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,         MANO3710
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,      MANO3720
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,        MANO3730
     *VARYX,XO,XBAR,YO,YBAR                                             MANO3740
      DOUBLE PRECISION ALPH2,ALPHAT,B,B2,BETH2,BETHAT,                  MANO3750
     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,PRINT,QUAN1,                  MANO3760
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,         MANO3770
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,      MANO3780
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,        MANO3790
     *VARYX,XO,XBAR,YO,YBAR                                             MANO3800
      DOUBLE PRECISION AVGX,AVGY,SDX,SDY                                MANO3810
      REAL X(N),Y(N)                                                    MANO3820
      WRITE(6,100) VARXY, RHO                                           MANO3830
100   FORMAT(///,1X,'LINEAR REGRESSION USING MANDEL PROCEDURE FOR ',    MANO3840
     &'ERROR IN THE INDEPENDENT VARIABLE',/,1X,78('='),////,T11,        MANO3850
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     &'PREDETERMINED RATIO VAR(X-ERROR)/VAR(Y-ERROR) = ',F8.3,////,     MANO3860
     &T11,'PREDETERMINED CORRELATION OF X-ERROR WITH Y-ERROR = ',F6.3)  MANO3870
      CALL STAT(X,N,AVGX,SDX,XMIN,XMAX,XSKEW,XKURT)                     MANO3880
      CALL STAT(Y,N,AVGY,SDY,YMIN,YMAX,YSKEW,YKURT)                     MANO3890
      WRITE(6,200) N, XMIN, YMIN, XMAX, YMAX, AVGX, AVGY, SDX, SDY,     MANO3900
     &XSKEW, YSKEW, XKURT, YKURT                                        MANO3910
200   FORMAT(///,T44,I4,' DATA POINTS',/,T36,33('-'),/,T43,'X',         MANO3920
     &T62,'Y',/,T36,14('-'),T55,14('-'),/,T26,'MINIMUM',T36,F14.6,      MANO3930
     &T55,F14.6,/,T26,'MAXIMUM',T36,F14.6,T55,F14.6,/,T29,'MEAN',       MANO3940
     &T36,F14.6,T55,F14.6,/,T15,'STANDARD DEVIATION',T36,F14.6,T55,     MANO3950
     &F14.6,/,T29,'SKEW',T40,F6.2,T59,F6.2,/,T25,'KURTOSIS',T40,F6.2,   MANO3960
     &T59,F6.2)                                                         MANO3970
      WRITE(6,300) ALPHAT, ALPH2, BETHAT, BETH2, STDEPS, STDDLT, BK1,   MANO3980
     *BK2,STDALP,STDAL2,STDBET,STDB2                                    MANO3990
300   FORMAT(///,T43,'PARAMETER ESTIMATES',/,T36,33('-'),/,T39,'Y = ',  MANO4000
     &'F(X)',T58,'X = F(Y)',/,T36,14('-'),T55,14('-'),/,T24,            MANO4010
     &'INTERCEPT',T36,F14.6,T55,F14.6,/,T28,'SLOPE',T36,F14.6,T55,      MANO4020
     &F14.6,/,1X,'RESIDUAL ERROR, S(YX) AND S(XY)',T36,F14.6,T55,F14.6, MANO4030
     &/,T12,'MANDEL''S BK STATISTIC',T40,F6.2,T59,F6.2,/,T6,'STANDARD ',MANO4040
     &'ERROR OF INTERCEPT',T36,F14.6,T55,F14.6,/,T10,'STANDARD ERROR ', MANO4050
     &'OF SLOPE',T36,F14.6,T55,F14.6)                                   MANO4060
      TALP1=-DABS(ALPHAT)/STDALP                                        MANO4070
      CALL TTABLE(N-2,TALP1,SIGAL1,1)                                   MANO4080
      TALPH2=-DABS(ALPH2)/STDAL2                                        MANO4090
      CALL TTABLE(N-2,TALPH2,SIGAL2,1)                                  MANO4100
      TBETH1=-DABS(BETHAT)/STDBET                                       MANO4110
      CALL TTABLE(N-2,TBETH1,SIGBH1,1)                                  MANO4120
      TBETH2=-DABS(BETH2)/STDB2                                         MANO4130
      CALL TTABLE(N-2,TBETH2,SIGBH2,1)                                  MANO4140
      WRITE(6,400) SIGAL1, SIGAL2, SIGBH1, SIGBH2                       MANO4150
400   FORMAT(///,T37,'APPROXIMATE SIGNIFICANCE LEVELS',/,T36,33('-'),/, MANO4160
     &T17,'NULL HYPOTHESIS',T38,'Y = F(X)',T59,'X = F(Y)',/,T16,17('-'),MANO4170
     &T36,12('-'),T57,12('-'),/,T16,'INTERCEPT IS ZERO',T40,F5.3,T61,   MANO4180
     &F5.3,/,T20,'SLOPE IS ZERO',T40,F5.3,T61,F5.3)                     MANO4190
1     WRITE(6,500)                                                      MANO4200
500   FORMAT(///,T2,'SELECT ONE OF THE FOLLOWING OPTIONS',/,T5,'1.  ',  MANO4210
     &'INTERVAL ESTIMATES FOR INTERCEPT, SLOPE, AND RESIDUAL ERROR',    MANO4220
     &/,T5,'2.  INTERVAL ESTIMATES FOR TRUE Y AT SELECTED VALUE OF X',/ MANO4230
     &,T5,'3.  INTERVAL ESTIMATES FOR TRUE X AT SELECTED VALUE OF Y',/, MANO4240
     &T5,'4.  TERMINATE THIS RUN')                                      MANO4250
      READ(5,*) INPUT                                                   MANO4260
      IF(INPUT.EQ.1) CALL CONF1(N)                                      MANO4270
      IF((INPUT.EQ.2).OR.(INPUT.EQ.3)) CALL CONF2(N,INPUT)              MANO4280
      IF((INPUT.LE.0).OR.(INPUT.GE.4)) RETURN                           MANO4290
      GO TO 1                                                           MANO4300
      END                                                               MANO4310
C                                                                       MANO4320
      SUBROUTINE CONF1(N)                                               MANO4330
      COMMON ALPH2,ALPHAT,B,B2,BETH2,BETHAT,BK1,BK2,                    MANO4340
     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,QUAN1,                        MANO4350
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,         MANO4360
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,      MANO4370
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,        MANO4380
     *VARYX,XO,XBAR,YO,YBAR                                             MANO4390
      DOUBLE PRECISION ALPH2,ALPHAT,B,B2,BETH2,BETHAT,                  MANO4400
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     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,PRINT,QUAN1,
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,
     *VARYX,XO,XBAR,YO,YBAR
      DOUBLE PRECISION AL1MIN,AL2MAX,AL2MIN,ALPMAX,B2MAX,B2MIN,
     *BETMAX,BETMIN,SDMAX,SDMIN,SEPMIN
      REAL C(10),SIGNIF(10),T(10)
      DATA SIGNIF/.005,.01,.025,.05,.1,.9,.95,.975,.99,.995/
      WRITE(6,100)
100   FORMAT(///,T18,'APPROXIMATE INTERVAL ESTIMATES FOR INTERCEPT',/,
     &T16,49('-'),/,T17,'TAIL AREA',T35,'Y = F(X)',T54,'X = F(Y)',
     &/,T16,11('-'),T32,14('-'),T51,14('-'))
      DO 1 I=1,5
      CALL TTABLE(N-2,T(I),1.-SIGNIF(I),2)
      AL1MIN=ALPHAT-T(I)*STDALP
      AL2MIN=ALPH2-T(I)*STDAL2
1     WRITE(6,200) SIGNIF(I), AL1MIN, AL2MIN
200   FORMAT(T16,'LOWER ',F5.3,T32,F14.6,T51,F14.6)
      DO 2 I=1,5
      J=6-I
      ALPMAX=ALPHAT+T(J)*STDALP
      AL2MAX=ALPH2+T(J)*STDAL2
2     WRITE(6,300) SIGNIF(J), ALPMAX, AL2MAX
300   FORMAT(T16,'UPPER ',F5.3,T32,F14.6,T51,F14.6)
      WRITE(6,400)
400   FORMAT(///,T20,'APPROXIMATE INTERVAL ESTIMATES FOR SLOPE',/,T16,
     &49('-'),/,T17,'TAIL AREA',T35,'Y = F(X)',T54,'X = F(Y)',/,T16,
     &11('-'),T32,14('-'),T51,14('-'))
      DO 3 I=1,5
      BETMIN=BETHAT-T(I)*STDBET
      B2MIN=BETH2-T(I)*STDB2
3     WRITE(6,200) SIGNIF(I), BETMIN, B2MIN
      DO 4 I=1,5
      J=6-I
      BETMAX=BETHAT+T(J)*STDBET
      B2MAX=BETH2+T(J)*STDB2
4     WRITE(6,300) SIGNIF(J), BETMAX, B2MAX
      WRITE(6,500)
500   FORMAT(///,T16,'APPROXIMATE INTERVAL ESTIMATES FOR RESIDUAL ERROR'
     &,/,T16,49('-'),/,T17,'TAIL AREA',T37,'S(YX)',T56,'S(XY)',
     &/,T16,11('-'),T32,14('-'),T51,14('-'))
      DO 5 I=1,5
      CALL CSTABL(N-2,CHISQ,1.-SIGNIF(I),2)
      SEPMIN=STDEPS/SQRT(CHISQ/FLOAT(N-2))
      SDMIN=STDDLT/SQRT(CHISQ/FLOAT(N-2))
5     WRITE(6,200) SIGNIF(I), SEPMIN, SDMIN
      DO 6 I=1,5
      J=6-I
      CALL CSTABL(N-2,CHISQ,SIGNIF(J),2)
      SEMAX=STDEPS/SQRT(CHISQ/FLOAT(N-2))
      SDMAX=STDDLT/SQRT(CHISQ/FLOAT(N-2))
6     WRITE(6,300) SIGNIF(J), SEMAX, SDMAX
      RETURN
      END
```

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C                                                                       MANO4960
      SUBROUTINE CONF2(N,INPUT)                                         MANO4970
      COMMON ALPH2,ALPHAT,B,B2,BETH2,BETHAT,BK1,BK2,                    MANO4980
     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,QUAN1,                        MANO4990
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,         MANO5000
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,      MANO5010
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,        MANO5020
     *VARYX,XO,XBAR,YO,YBAR                                             MANO5030
      DOUBLE PRECISION ALPH2,ALPHAT,B,B2,BETH2,BETHAT,                  MANO5040
     *C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,K,K2,PRINT,QUAN1,                  MANO5050
     *QUAN2,QUAN3,QUAN4,QUAN5,QUAN6,SUMX,SUMY,SSUU,SSUU2,SSVV2,         MANO5060
     *SSVV,SSXX,SSXY,SSYY,STDAL2,STDALP,STDB2,STDBET,STDD2,STDDLT,      MANO5070
     *STDE,STDE2,STDEP2,STDEPS,STDVXO,STDVYO,THETA,THETA2,VARXY,        MANO5080
     *VARYX,XO,XBAR,YO,YBAR                                             MANO5090
      DOUBLE PRECISION XLIM(10),YLIM(10)                                MANO5100
      REAL BK1,BK2,SIGNIF(10),T(10)                                     MANO5110
      DATA SIGNIF/.005,.01,.025,.05,.1,.9,.95,.975,.99,.995/            MANO5120
      DO 10 I=1,10                                                      MANO5130
10    CALL TTABLE(N-2,T(I),SIGNIF(I),2)                                 MANO5140
      IF(INPUT.EQ.3)GO TO 40                                            MANO5150
      WRITE(6,100)                                                      MANO5160
100   FORMAT(/,1X,'ENTER VALUE OF X AT WHICH COMPUTATIONS ARE TO BE ',  MANO5170
     &'MADE')                                                           MANO5180
      READ(5,*)XO                                                       MANO5190
      YO=ALPHAT+BETHAT*XO                                               MANO5200
      C5=(XO-XBAR)**2                                                   MANO5210
      STDVYO=DSQRT(C3+C4*C5)                                            MANO5220
      WRITE(6,200) YO, STDVYO                                           MANO5230
200   FORMAT(/,T21,'POINT ESTIMATE OF TRUE Y = ',F14.6,///,T21,'S(Y ',  MANO5240
     &'FOR OBSERVED X)      = ',F14.6)                                  MANO5250
      WRITE(6,300)                                                      MANO5260
300   FORMAT(//,T27,'APPROXIMATE INTERVAL ESTIMATES',/,T26,32('-'),/,   MANO5270
     &T27,'TAIL AREA',T42,'Y FOR OBSERVED X',/,T26,11('-'),T42,16('-')) MANO5280
      DO 20 I=1,5                                                       MANO5290
      YLIM(I)=YO+T(I)*STDVYO                                            MANO5300
      WRITE(6,400) SIGNIF(I),YLIM(I)                                    MANO5310
400   FORMAT(T26,'LOWER ',F5.3,T44,F14.6)                               MANO5320
      J=6                                                               MANO5330
20    CONTINUE                                                          MANO5340
      DO 30 I=6,10                                                      MANO5350
      J=J-1                                                             MANO5360
      YLIM(I)=YO+T(I)*STDVYO                                            MANO5370
      WRITE(6,500)SIGNIF(J),YLIM(I)                                     MANO5380
500   FORMAT(T26,'UPPER ',F5.3,T44,F14.6)                               MANO5390
30    CONTINUE                                                          MANO5400
      RETURN                                                            MANO5410
40    CONTINUE                                                          MANO5420
      WRITE(6,600)                                                      MANO5430
600   FORMAT(/,1X,'ENTER VALUE OF Y AT WHICH COMPUTATIONS ARE TO BE ',  MANO5440
     &'MADE')                                                           MANO5450
      READ(5,*)YO                                                       MANO5460
      XO=ALPH2+BETH2*YO                                                 MANO5470
      C10=(YO-YBAR)**2                                                  MANO5480
      STDVXO=DSQRT(C8+C9*C10)                                           MANO5490
      WRITE(6,700) XO, STDVXO                                           MANO5500
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700   FORMAT(/,T21,'POINT ESTIMATE OF TRUE X = ',F14.6,///,T21,'S(X ',
     &'FOR OBSERVED Y)       = ',F14.6)
      WRITE(6,800)
800   FORMAT(//,T27,'APPROXIMATE INTERVAL ESTIMATES',/,T26,32('-'),/,
     &T27,'TAIL AREA',T42,'X FOR OBSERVED Y',/,T26,11('-'),T42,16('-'))
      DO 50 I=1,5
      XLIM(I)=XO+T(I)*STDVXO
      WRITE(6,400)SIGNIF(I),XLIM(I)
50    CONTINUE
      J=6
      DO 60 I=6,10
      J=J-1
      XLIM(I)=XO+T(I)*STDVXO
      WRITE(6,500)SIGNIF(J),XLIM(I)
60    CONTINUE
      RETURN
      END
C
      SUBROUTINE CSTABL(NDF,CHISQ,AREA,MODE)
      IF(NDF.LT.1) GO TO 3
      IF(MODE.NE.1) GO TO 2
      IF(CHISQ.LT.0.) GO TO 4
      IF(NDF.GT.200) GO TO 1
      AREA=AC(NDF,CHISQ)
      RETURN
1     X=2./(9.*NDF)
      Z=((CHISQ/NDF)**.3333333-1.+X)/SQRT(X)
      CALL ZTABLE(Z,AREA,1)
      RETURN
2     IF(MODE.NE.2) GO TO 5
      IF((AREA.LT.0.0001).OR.(AREA.GT.0.9999)) GO TO 6
      X=2./(9.*NDF)
      CALL ZTABLE(Z,AREA,2)
      CHISQ=AMAX1(0.,NDF*(1.-X+Z*SQRT(X))**3)
      IF(NDF.GT.200) RETURN
      CHISQ=CA(NDF,AREA,CHISQ)
      RETURN
3     WRITE(6,100)
100   FORMAT('0','DEGREES OF FREEDOM IS LESS THAN ONE IN SUBROUTINE ',
     &'CSTABL')
      STOP
4     WRITE(6,200)
200   FORMAT('0','CHI-SQUARE VALUE IS NOT SPECIFIED CORRECTLY IN ',
     &'SUBROUTINE CSTABL')
      STOP
5     WRITE(6,300)
300   FORMAT('0','MODE OF OPERATION IS NOT SPECIFIED CORRECTLY IN ',
     &'SUBROUTINE CSTABL')
      STOP
6     WRITE(6,400)
400   FORMAT('0','AREA IS OUTSIDE LIMITS OF 0.0001 - 0.9999 IN ',
     &'SUBROUTINE CSTABL')
      STOP
      END
C
```

```
      FUNCTION AC(NDF,CHISQ)                                            MANO06060
      DF=NDF                                                            MANO06070
      CALL CDTR(CHISQ,DF,AC,D,IER)                                      MANO06080
      IF(IER.EQ.0) RETURN                                               MANO06090
      WRITE(6,100)                                                      MANO06100
100   FORMAT('0','ERROR IN CHI-SQUARE AREA CALCULATION IN SUBROUTINE ', MANO06110
     &'CSTABL')                                                         MANO06120
      STOP                                                              MANO06130
      END                                                               MANO06140
C                                                                       MANO06150
      FUNCTION CA(NDF,AREA,CSTART)                                      MANO06160
      A(C)=AC(NDF,C)                                                    MANO06170
      CA=CSTART                                                         MANO06180
      ATRIAL=A(CA)                                                      MANO06190
      KOUNT=0                                                           MANO06200
1     KOUNT=KOUNT+1                                                     MANO06210
      IF(KOUNT.GT.100) GO TO 3                                          MANO06220
      ERROR=ATRIAL-AREA                                                 MANO06230
      IF(ABS(ERROR).LE.0.00001) RETURN                                  MANO06240
      IF(CA.LE.0.) CA=1.E-6                                             MANO06250
      C1=CA                                                             MANO06260
      A1=ATRIAL                                                         MANO06270
      C2=C1                                                             MANO06280
2     C2=1.001*C2                                                       MANO06290
      IF(C2/C1.GT.1.01) GO TO 3                                         MANO06300
      A2=A(C2)                                                          MANO06310
      IF(A1.EQ.A2) GO TO 2                                              MANO06320
      SLOPE=(A2-A1)/(C2-C1)                                             MANO06330
      CA=C2+(AREA-A2)/SLOPE                                             MANO06340
      ATRIAL=A(CA)                                                      MANO06350
      GO TO 1                                                           MANO06360
3     WRITE(6,100)                                                      MANO06370
100   FORMAT('0','EXCESSIVE CALCULATION TIME REQUIRED IN SUBROUTINE ',  MANO06380
     &'CSTABL')                                                         MANO06390
      STOP                                                              MANO06400
      END                                                               MANO06410
C                                                                       MANO06420
      SUBROUTINE ZTABLE(Z,AREA,MODE)                                    MANO06430
      DATA A1/.33267/, B1/.4361836/, B2/-.1201676/, B3/.937298/,        MANO06440
     &C1/2.515517/, C2/.802853/, C3/.010328/, D1/1.432788/,             MANO06450
     &D2/.189269/, D3/.001308/, SQRT2P/2.506628/                        MANO06460
      IF(MODE.NE.1) GO TO 2                                             MANO06470
      AREA=1.                                                           MANO06480
      ZABS=ABS(Z)                                                       MANO06490
      IF(ZABS.GT.5.) GO TO 1                                            MANO06500
      X=1./(1.+A1*ZABS)                                                 MANO06510
      Y=EXP(-ZABS**2/2)/SQRT2P                                          MANO06520
      AREA=1.-Y*X*(B1+X*(B2+X*B3))                                      MANO06530
1     IF(Z.GE.0.) RETURN                                                MANO06540
      AREA=1.-AREA                                                      MANO06550
      RETURN                                                            MANO06560
2     IF(MODE.NE.2) GO TO 4                                             MANO06570
      IF((AREA.LT.0.).OR.(AREA.GT.1.)) GO TO 5                          MANO06580
      Z=5.                                                              MANO06590
      A=AREA                                                            MANO06600
```

```
      IF(A.GT.0.5) A=1.-A                                               MAN06610
      IF(A.LT.0.0000004) GO TO 3                                        MAN06620
      X=SQRT(ALOG(1./A**2))                                             MAN06630
      Z=X-(C1+X*(C2+X*C3))/(1.+X*(D1+X*(D2+X*D3)))                      MAN06640
3     IF(AREA.GE.0.5) RETURN                                            MAN06650
      Z=-Z                                                              MAN06660
      RETURN                                                            MAN06670
4     WRITE(6,100)                                                      MAN06680
100   FORMAT('0','MODE OF OPERATION IS NOT SPECIFIED CORRECTLY IN ',    MAN06690
     &'SUBROUTINE ZTABLE')                                              MAN06700
      STOP                                                              MAN06710
5     WRITE(6,200)                                                      MAN06720
200   FORMAT('0','AREA IS NOT SPECIFIED CORRECTLY IN SUBROUTINE ZTABLE')MAN06730
      STOP                                                              MAN06740
      END                                                               MAN06750
C                                                                       MAN06760
C     ..................................................................MAN06770
C                                                                       MAN06780
C        SUBROUTINE CDTR                                                MAN06790
C                                                                       MAN06800
C        PURPOSE                                                        MAN06810
C           COMPUTES P(X) = PROBABILITY THAT THE RANDOM VARIABLE U,     MAN06820
C           DISTRIBUTED ACCORDING TO THE CHI-SQUARE DISTRIBUTION WITH G MAN06830
C           DEGREES OF FREEDOM, IS LESS THAN OR EQUAL TO X.  F(G,X), THEMAN06840
C           ORDINATE OF THE CHI-SQUARE DENSITY AT X, IS ALSO COMPUTED.  MAN06850
C                                                                       MAN06860
C        USAGE                                                          MAN06870
C           CALL CDTR(X,G,P,D,IER)                                      MAN06880
C                                                                       MAN06890
C        DESCRIPTION OF PARAMETERS                                      MAN06900
C           X   - INPUT SCALAR FOR WHICH P(X) IS COMPUTED.              MAN06910
C           G   - NUMBER OF DEGREES OF FREEDOM OF THE CHI-SQUARE        MAN06920
C                 DISTRIBUTION.  G IS A CONTINUOUS PARAMETER.           MAN06930
C           P   - OUTPUT PROBABILITY.                                   MAN06940
C           D   - OUTPUT DENSITY.                                       MAN06950
C           IER - RESULTANT ERROR CODE WHERE                            MAN06960
C               IER= 0 --- NO ERROR                                     MAN06970
C               IER=-1 --- AN INPUT PARAMETER IS INVALID.  X IS LESS    MAN06980
C                          THAN 0.0, OR G IS LESS THAN 0.5 OR GREATER   MAN06990
C                          THAN 2*10**(+5).  P AND D ARE SET TO -1.E75.MAN07000
C               IER=+1 --- INVALID OUTPUT.  P IS LESS THAN ZERO OR      MAN07010
C                          GREATER THAN ONE, OR SERIES FOR T1 (SEE      MAN07020
C                          MATHEMATICAL DESCRIPTION) HAS FAILED TO      MAN07030
C                          CONVERGE.  P IS SET TO 1.E75.                MAN07040
C                                                                       MAN07050
C        REMARKS                                                        MAN07060
C           SEE MATHEMATICAL DESCRIPTION.                               MAN07070
C                                                                       MAN07080
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  MAN07090
C           DLGAM                                                       MAN07100
C           NDTR                                                        MAN07110
C                                                                       MAN07120
C        METHOD                                                         MAN07130
C           REFER TO R.E. BARGMANN AND S.P. GHOSH, STATISTICAL          MAN07140
C           DISTRIBUTION PROGRAMS FOR A COMPUTER LANGUAGE,              MAN07150
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C           IBM RESEARCH REPORT RC-1094, 1963.
C
C     .............................................................
C
      SUBROUTINE CDTR(X,G,P,D,IER)
      DOUBLE PRECISION XX,DLXX,X2,DLX2,GG,G2,DLT3,THETA,THP1,
     1GLG2,DD,T11,SER,CC,XI,FAC,TLOG,TERM,GTH,A2,A,B,C,DT2,DT3,THPI
C
C        TEST FOR VALID INPUT DATA
C
      IF(G-(.5-1.E-5)) 590,10,10
   10 IF(G-2.E+5) 20,20,590
   20 IF(X) 590,30,30
C
C        TEST FOR X NEAR 0.0
C
   30 IF(X-1.E-8) 40,40,80
   40 P=0.0
      IF(G-2.) 50,60,70
   50 D=1.E75
      GO TO 610
   60 D=0.5
      GO TO 610
   70 D=0.0
      GO TO 610
C
C        TEST FOR X GREATER THAN 1.E+6
C
   80 IF(X-1.E+6) 100,100,90
   90 D=0.0
      P=1.0
      GO TO 610
C
C        SET PROGRAM PARAMETERS
C
  100 XX=DBLE(X)
      DLXX=DLOG(XX)
      X2=XX/2.D0
      DLX2=DLOG(X2)
      GG=DBLE(G)
      G2=GG/2.D0
C
C        COMPUTE ORDINATE
C
      CALL DLGAM(G2,GLG2,IOK)
      DD=(G2-1.D0)*DLXX-X2-G2*.6931471805599453 -GLG2
      IF(DD-1.68D02) 110,110,120
  110 IF(DD+1.68D02) 130,130,140
  120 D=1.E75
      GO TO 150
  130 D=0.0
      GO TO 150
  140 DD=DEXP(DD)
      D=SNGL(DD)
C
```

```
C         TEST FOR G GREATER THAN 1000.0
C         TEST FOR X GREATER THAN 2000.0
C
  150 IF(G-1000.) 160,160,180
  160 IF(X-2000.) 190,190,170
  170 P=1.0
      GO TO 610
  180 A=DLOG(XX/GG)/3.D0
      A=DEXP(A)
      B=2.D0/(9.D0*GG)
      C=(A-1.D0+B)/DSQRT(B)
      SC=SNGL(C)
      CALL NDTR(SC,P,DUMMY)
      GO TO 490
C
C         COMPUTE THETA
C
  190 K= IDINT(G2)
      THETA=G2-DFLOAT(K)
      IF(THETA-1.D-8) 200,200,210
  200 THETA=0.D0
  210 THP1=THETA+1.D0
C
C         SELECT METHOD OF COMPUTING T1
C
      IF(THETA) 230,230,220
  220 IF(XX-10.D0) 260,260,320
C
C         COMPUTE T1 FOR THETA EQUALS 0.0
C
  230 IF(X2-1.68D02) 250,240,240
  240 T1=1.0
      GO TO 400
  250 T11=1.D0-DEXP(-X2)
      T1=SNGL(T11)
      GO TO 400
C
C         COMPUTE T1 FOR THETA GREATER THAN 0.0 AND
C         X LESS THAN OR EQUAL TO 10.0
C
  260 SER=X2*(1.D0/THP1 -X2/(THP1+1.D0))
      J=+1
      CC=DFLOAT(J)
      DO 270 IT1=3,30
      XI=DFLOAT(IT1)
      CALL DLGAM(XI,FAC,IOK)
      TLOG= XI*DLX2-FAC-DLOG(XI+THETA)
      TERM=DEXP(TLOG)
      TERM=DSIGN(TERM,CC)
      SER=SER+TERM
      CC=-CC
      IF(DABS(TERM)-1.D-9) 280,270,270
  270 CONTINUE
      GO TO 600
  280 IF(SER) 600,600,290
```

```
  290 CALL DLGAM(THP1,GTH,IOK)                                          MAN08260
      TLOG=THETA*DLX2+DLOG(SER)-GTH                                     MAN08270
      IF(TLOG+1.68D02) 300,300,310                                      MAN08280
  300 T1=0.0                                                            MAN08290
      GO TO 400                                                         MAN08300
  310 T11=DEXP(TLOG)                                                    MAN08310
      T1=SNGL(T11)                                                      MAN08320
      GO TO 400                                                         MAN08330
C                                                                       MAN08340
C         COMPUTE T1 FOR THETA GREATER THAN 0.0 AND                     MAN08350
C         X GREATER THAN 10.0 AND LESS THAN 2000.0                      MAN08360
C                                                                       MAN08370
  320 A2=0.D0                                                           MAN08380
      DO 340 I=1,25                                                     MAN08390
      XI=DFLOAT(I)                                                      MAN08400
      CALL DLGAM(THP1,GTH,IOK)                                          MAN08410
      T11=-(13.D0*XX)/XI +THP1*DLOG(13.D0*XX/XI) -GTH-DLOG(XI)          MAN08420
      IF(T11+1.68D02) 340,340,330                                       MAN08430
  330 T11=DEXP(T11)                                                     MAN08440
      A2=A2+T11                                                         MAN08450
  340 CONTINUE                                                          MAN08460
      A=1.01282051+THETA/156.D0-XX/312.D0                               MAN08470
      B=DABS(A)                                                         MAN08480
      C= -X2+THP1*DLX2+DLOG(B)-GTH-3.95124371858 1427                   MAN08490
      IF(C+1.68D02) 370,370,350                                         MAN08500
  350 IF (A) 360,370,380                                                MAN08510
  360 C=-DEXP(C)                                                        MAN08520
      GO TO 390                                                         MAN08530
  370 C=0.D0                                                            MAN08540
      GO TO 390                                                         MAN08550
  380 C=DEXP(C)                                                         MAN08560
  390 C=A2+C                                                            MAN08570
      T11=1.D0-C                                                        MAN08580
      T1=SNGL(T11)                                                      MAN08590
C                                                                       MAN08600
C         SELECT PROPER EXPRESSION FOR P                                MAN08610
C                                                                       MAN08620
  400 IF(G-2.) 420,410,410                                              MAN08630
  410 IF(G-4.) 450,460,460                                              MAN08640
C                                                                       MAN08650
C         COMPUTE P FOR G GREATER THAN ZERO AND LESS THAN 2.0           MAN08660
C                                                                       MAN08670
  420 CALL DLGAM(THP1,GTH,IOK)                                          MAN08680
      DT2=THETA*DLXX-X2-THP1*.6931471805599453 -GTH                     MAN08690
      IF(DT2+1.68D02) 430,430,440                                       MAN08700
  430 P=T1                                                              MAN08710
      GO TO 490                                                         MAN08720
  440 DT2=DEXP(DT2)                                                     MAN08730
      T2=SNGL(DT2)                                                      MAN08740
      P=T1+T2+T2                                                        MAN08750
      GO TO 490                                                         MAN08760
C                                                                       MAN08770
C         COMPUTE P FOR G GREATER THAN OR EQUAL TO 2.0                  MAN08780
C         AND LESS THAN 4.0                                             MAN08790
C                                                                       MAN08800
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  450 P=T1                                                              MAN08810
      GO TO 490                                                         MAN08820
C                                                                       MAN08830
C         COMPUTE P FOR G GREATER THAN OR EQUAL TO 4.0                  MAN08840
C         AND LESS THAN OR EQUAL TO 1000.0                              MAN08850
C                                                                       MAN08860
  460 DT3=0.D0                                                          MAN08870
      DO 480 I3=2,K                                                     MAN08880
      THPI=DFLOAT(I3)+THETA                                             MAN08890
      CALL DLGAM(THPI,GTH,IOK)                                          MAN08900
      DLT3=THPI•DLX2-DLXX-X2-GTH                                        MAN08910
      IF(DLT3+1.68D02) 480,480,470                                      MAN08920
  470 DT3=DT3+DEXP(DLT3)                                                MAN08930
  480 CONTINUE                                                          MAN08940
      T3=SNGL(DT3)                                                      MAN08950
      P=T1-T3-T3                                                        MAN08960
C                                                                       MAN08970
C         SET ERROR INDICATOR                                           MAN08980
C                                                                       MAN08990
  490 IF(P) 500,520,520                                                 MAN09000
  500 IF(ABS(P)-1.E-7) 510,510,600                                      MAN09010
  510 P=0.0                                                             MAN09020
      GO TO 610                                                         MAN09030
  520 IF(1.-P) 530,550,550                                              MAN09040
  530 IF(ABS(1.-P)-1.E-7) 540,540,600                                   MAN09050
  540 P=1.0                                                             MAN09060
      GO TO 610                                                         MAN09070
  550 IF(P-1.E-8) 560,560,570                                           MAN09080
  560 P=0.0                                                             MAN09090
      GO TO 610                                                         MAN09100
  570 IF((1.0-P)-1.E-8) 580,580,610                                     MAN09110
  580 P=1.0                                                             MAN09120
      GO TO 610                                                         MAN09130
  590 IER=-1                                                            MAN09140
      D=-1.E75                                                          MAN09150
      P=-1.E75                                                          MAN09160
      GO TO 620                                                         MAN09170
  600 IER=+1                                                            MAN09180
      P= 1.E75                                                          MAN09190
      GO TO 620                                                         MAN09200
  610 IER=0                                                             MAN09210
  620 RETURN                                                            MAN09220
      END                                                               MAN09230
C                                                                       MAN09240
C         SUBROUTINE DLGAM                                              MAN09250
C                                                                       MAN09260
C         PURPOSE                                                       MAN09270
C            COMPUTES THE DOUBLE PRECISION NATURAL LOGARITHM OF THE     MAN09280
C            GAMMA FUNCTION OF A GIVEN DOUBLE PRECISION ARGUMENT.       MAN09290
C                                                                       MAN09300
C         USAGE                                                         MAN09310
C            CALL DLGAM(XX,DLNG,IER)                                    MAN09320
C                                                                       MAN09330
C         DESCRIPTION OF PARAMETERS                                     MAN09340
C            XX    - THE DOUBLE PRECISION ARGUMENT FOR THE LOG GAMMA    MAN09350
```

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C                  FUNCTION.
C           DLNG - THE RESULTANT DOUBLE PRECISION LOG GAMMA FUNCTION
C                  VALUE.
C           IER  - RESULTANT ERROR CODE WHERE
C                  IER= 0----NO ERROR.
C                  IER=-1----XX IS WITHIN 10**(-9) OF BEING ZERO OR XX
C                            IS NEGATIVE.  DLNG IS SET TO -1.0D75.
C                  IER=+1----XX IS GREATER THAN 10**70. DLNG IS SET TO
C                            +1.0D75.
C
C        REMARKS
C           NONE
C
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           THE EULER-MCLAURIN EXPANSION TO THE SEVENTH DERIVATIVE TERM
C           IS USED, AS GIVEN BY M. ABRAMOWITZ AND I.A. STEGUN,
C           'HANDBOOK OF MATHEMATICAL FUNCTIONS', U. S. DEPARTMENT OF
C           COMMERCE, NATIONAL BUREAU OF STANDARDS APPLIED MATHEMATICS
C           SERIES, 1966, EQUATION 6.1.41.
C
C     ..................................................................
C
      SUBROUTINE DLGAM(XX,DLNG,IER)
      DOUBLE PRECISION XX,ZZ,TERM,RZ2,DLNG
      IER=0
      ZZ=XX
      IF(XX-1.D10) 2,2,1
    1 IF(XX-1.D70) 8,9,9
C
C        SEE IF XX IS NEAR ZERO OR NEGATIVE
C
    2 IF(XX-1.D-9) 3,3,4
    3 IER=-1
      DLNG=-1.D75
      GO TO 10
C
C        XX GREATER THAN ZERO AND LESS THAN OR EQUAL TO 1.D+10
C
    4 TERM=1.D0
    5 IF(ZZ-18.D0) 6,6,7
    6 TERM=TERM*ZZ
      ZZ=ZZ+1.D0
      GO TO 5
    7 RZ2=1.D0/ZZ**2
      DLNG =(ZZ-0.5D0)*DLOG(ZZ)-ZZ +0.9189385332046727 -DLOG(TERM)+
     1(1.D0/ZZ)*(.8333333333333333D-1 -(RZ2*(.2777777777777777D-2 +(RZ2*
     2(.7936507936507936D-3 -(RZ2*(.5952380952380952D-3)))))))
      GO TO 10
C
C        XX GREATER THAN 1.D+10 AND LESS THAN 1.D+70
C
    8 DLNG=ZZ*(DLOG(ZZ)-1.D0)
```

```
      GO TO 10
C
C        XX GREATER THAN OR EQUAL TO 1.D+70
C
    9 IER=+1
      DENG=1.D75
   10 RETURN
      END
C
C.......................................................................
C
C        SUBROUTINE NDTR
C
C        PURPOSE
C           COMPUTES Y = P(X) = PROBABILITY THAT THE RANDOM VARIABLE  U,
C           DISTRIBUTED NORMALLY(0,1), IS LESS THAN OR EQUAL TO X.
C           F(X), THE ORDINATE OF THE NORMAL DENSITY AT X, IS ALSO
C           COMPUTED.
C
C        USAGE
C           CALL NDTR(X,P,D)
C
C        DESCRIPTION OF PARAMETERS
C           X--INPUT SCALAR FOR WHICH P(X) IS COMPUTED.
C           P--OUTPUT PROBABILITY.
C           D--OUTPUT DENSITY.
C
C        REMARKS
C           MAXIMUM ERROR IS 0.0000007.
C
C        SUBROUTINES AND SUBPROGRAMS REQUIRED
C           NONE
C
C        METHOD
C           BASED ON APPROXIMATIONS IN C. HASTINGS, APPROXIMATIONS FOR
C           DIGITAL COMPUTERS, PRINCETON UNIV. PRESS, PRINCETON, N.J.,
C           1955.  SEE EQUATION 26.2.17, HANDBOOK OF MATHEMATICAL
C           FUNCTIONS, ABRAMOWITZ AND STEGUN, DOVER PUBLICATIONS, INC.,
C           NEW YORK.
C
C.......................................................................
C
      SUBROUTINE NDTR(X,P,D)
C
      AX=ABS(X)
      T=1.0/(1.0+.2316419*AX)
      D=0.3989423*EXP(-X*X/2.0)
      P = 1.0 - D*T*((((1.330274*T - 1.821256)*T + 1.781478)*T -
     1  0.3565638)*T + 0.3193815)
      IF(X)1,2,2
    1 P=1.0-P
    2 RETURN
      END
      SUBROUTINE TTABLE(NDF,T,AREA,MODE)
      IF(NDF.LT.1) GO TO 6
```

```
      IF(MODE.NE.1) GO TO 3                                             MAN10460
      IF(NDF.GE.200) GO TO 1                                            MAN10470
      IF(ABS(T).GT.(70.-FLOAT(NDF)/4.)) GO TO 2                         MAN10480
      AREA=AT(NDF,T)                                                    MAN10490
      RETURN                                                            MAN10500
1     IF(NDF.GE.1000) GO TO 2                                           MAN10510
      A90=AT(90,T)                                                      MAN10520
      A100=AT(100,T)                                                    MAN10530
      AREA=A100-900.*(A90-A100)*(.01-1./NDF)                            MAN10540
      AREA=AMAX1(0.,AMIN1(1.,AREA))                                     MAN10550
      RETURN                                                            MAN10560
2     AREA=AZ(T)                                                        MAN10570
      RETURN                                                            MAN10580
3     IF(MODE.NE.2) GO TO 7                                             MAN10590
      IF((AREA.LT.0.0001).OR.(AREA.GT.0.9999)) GO TO 8                  MAN10600
      IF(NDF.GE.200) GO TO 4                                            MAN10610
      T=TA(NDF,AREA)                                                    MAN10620
      RETURN                                                            MAN10630
4     A=AMAX1(AREA,(1.-AREA))                                           MAN10640
      IF((A.LE.0.99).AND.(NDF.GE.1000)) GO TO 5                         MAN10650
      IF((A.LE.0.999).AND.(NDF.GE.5000)) GO TO 5                        MAN10660
      T90=TA(90,AREA)                                                   MAN10670
      T100=TA(100,AREA)                                                 MAN10680
      T=T100-900.*(T90-T100)*(.01-1./NDF)                               MAN10690
      RETURN                                                            MAN10700
5     T=ZA(AREA)                                                        MAN10710
      RETURN                                                            MAN10720
6     WRITE(6,100)                                                      MAN10730
100   FORMAT('0','DEGREES OF FREEDOM IS LESS THAN ONE IN SUBROUTINE ',  MAN10740
     &'TTABLE')                                                         MAN10750
      STOP                                                              MAN10760
7     WRITE(6,200)                                                      MAN10770
200   FORMAT('0','MODE OF OPERATION IS NOT SPECIFIED CORRECTLY IN ',    MAN10780
     &'SUBROUTINE TTABLE')                                              MAN10790
      STOP                                                              MAN10800
8     WRITE(6,300)                                                      MAN10810
300   FORMAT('0','AREA IS OUTSIDE LIMITS OF 0.0001 - 0.9999 IN ',       MAN10820
     &'SUBROUTINE TTABLE')                                              MAN10830
      STOP                                                              MAN10840
      END                                                               MAN10850
      FUNCTION AT(NDF,T)                                                MAN10860
      DATA PI/3.141593/                                                 MAN10870
      X=ATAN(T/SQRT(FLOAT(NDF)))                                        MAN10880
      COSX=COS(X)                                                       MAN10890
      IF((NDF/2)*2.EQ.NDF) GO TO 1                                      MAN10900
      K=1                                                               MAN10910
      SUM=0.                                                            MAN10920
      IF(NDF.EQ.1) GO TO 4                                              MAN10930
      SUM=COSX                                                          MAN10940
      IF(NDF.EQ.3) GO TO 4                                              MAN10950
      GO TO 2                                                           MAN10960
1     K=0                                                               MAN10970
      SUM=1.                                                            MAN10980
      IF(NDF.EQ.2) GO TO 4                                              MAN10990
2     COEF=1.                                                           MAN11000
```

```
      ISTOP=NDF/2-1                                                     MAN11010
      DO 3 I=1,ISTOP                                                    MAN11020
      J=2*I+K                                                           MAN11030
      COEF=COEF*(J-1)/J                                                 MAN11040
3     SUM=SUM+COEF*COSX**J                                              MAN11050
4     AT=(1.+(K*2./PI)*(X+SUM*SIN(X))+(1-K)*SUM*SIN(X))/2.              MAN11060
      RETURN                                                            MAN11070
      END                                                               MAN11080
      FUNCTION TA(NDF,AREA)                                             MAN11090
      Z=ZA(AREA)                                                        MAN11100
      TA=Z+(Z+Z**3)/(4.*NDF)+(3.*Z+16.*Z**3+5.*Z**5)/(96.*NDF**2)       MAN11110
      ATRIAL=AT(NDF,TA)                                                 MAN11120
      KOUNT=0                                                           MAN11130
1     KOUNT=KOUNT+1                                                     MAN11140
      IF(KOUNT.GT.100) GO TO 3                                          MAN11150
      ERROR=ATRIAL-AREA                                                 MAN11160
      IF(ABS(ERROR).LE.0.0000005) RETURN                                MAN11170
      IF(TA.EQ.0.) TA=1.E-6                                             MAN11180
      T1=TA                                                             MAN11190
      A1=ATRIAL                                                         MAN11200
      T2=T1                                                             MAN11210
2     T2=1.001*T2                                                       MAN11220
      IF(T2/T1.GT.1.01) GO TO 3                                         MAN11230
      A2=AT(NDF,T2)                                                     MAN11240
      IF(A1.EQ.A2) GO TO 2                                              MAN11250
      SLOPE=(A2-A1)/(T2-T1)                                             MAN11260
      TA=T2+(AREA-A2)/SLOPE                                             MAN11270
      ATRIAL=AT(NDF,TA)                                                 MAN11280
      GO TO 1                                                           MAN11290
3     WRITE(6,100)                                                      MAN11300
100   FORMAT('0','EXCESSIVE CALCULATION TIME IN SUBROUTINE TTABLE')     MAN11310
      STOP                                                              MAN11320
      END                                                               MAN11330
      FUNCTION AZ(Z)                                                    MAN11340
      DATA A1/.33267/, B1/.4361836/, B2/-.1201676/, B3/.937298/,        MAN11350
     &SQRT2P/2.506628/                                                  MAN11360
      AZ=1.                                                             MAN11370
      ZABS=ABS(Z)                                                       MAN11380
      IF(ZABS.GT.5.) GO TO 1                                            MAN11390
      X=1./(1.+A1*ZABS)                                                 MAN11400
      Y=EXP(-ZABS**2/2)/SQRT2P                                          MAN11410
      AZ=1.-Y*X*(B1+X*(B2+X*B3))                                        MAN11420
1     IF(Z.GE.0.) RETURN                                                MAN11430
      AZ=1.-AZ                                                          MAN11440
      RETURN                                                            MAN11450
      END                                                               MAN11460
      FUNCTION ZA(AREA)                                                 MAN11470
      DATA C1/2.515517/, C2/.802853/, C3/.010328/, D1/1.432788/,        MAN11480
     &D2/.189269/, D3/.001308/                                          MAN11490
      A=AMIN1(AREA,(1.-AREA))                                           MAN11500
      X=SQRT(ALOG(1./A**2))                                             MAN11510
      ZA=X-(C1+X*(C2+X*C3))/(1.+X*(D1+X*(D2+X*D3)))                     MAN11520
      IF(AREA.GE.0.5) RETURN                                            MAN11530
      ZA=-ZA                                                            MAN11540
      RETURN                                                            MAN11550
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      END                                                               MAN11560
      SUBROUTINE STAT(A,N,AVG,STDV,AMIN,AMAX,SKEW,CURT)                 MAN11570
      DIMENSION A(N)                                                    MAN11580
      DOUBLE PRECISION AI, D2, D3, D4, DI, SDN, SUM                     MAN11590
      IF(N.LT.1) GO TO 3                                                MAN11600
      A1=A(1)                                                           MAN11610
      AMIN=A1                                                           MAN11620
      AMAX=A1                                                           MAN11630
      AVG=A1                                                            MAN11640
      STDV=0.                                                           MAN11650
      SKEW=0.                                                           MAN11660
      CURT=0.                                                           MAN11670
      IF(N.LT.2) GO TO 4                                                MAN11680
      SUM=0.                                                            MAN11690
      DO 1 I=1,N                                                        MAN11700
      AI=A(I)                                                           MAN11710
      IF(AI.LT.AMIN) AMIN=AI                                            MAN11720
      IF(AI.GT.AMAX) AMAX=AI                                            MAN11730
1     SUM=SUM+AI                                                        MAN11740
      AVG=SUM/N                                                         MAN11750
      D2=0.                                                             MAN11760
      D3=0.                                                             MAN11770
      D4=0.                                                             MAN11780
      DO 2 I=1,N                                                        MAN11790
      DI=A(I)-AVG                                                       MAN11800
      D2=D2+DI**2                                                       MAN11810
      D3=D3+DI**3                                                       MAN11820
2     D4=D4+DI**4                                                       MAN11830
      IF(D2.EQ.0.) GO TO 5                                              MAN11840
      STDV=DSQRT(D2/(N-1))                                              MAN11850
      SDN=DSQRT(D2/N)                                                   MAN11860
      SKEW=D3/(N*SDN**3)                                                MAN11870
      CURT=D4/(N*SDN**4)-3.                                             MAN11880
      RETURN                                                            MAN11890
3     WRITE(6,100)                                                      MAN11900
100   FORMAT('0','N IS LESS THAN ONE IN SUBROUTINE STAT')               MAN11910
      STOP                                                              MAN11920
4     WRITE(6,200)                                                      MAN11930
200   FORMAT('0','***WARNING***',/,1X,'N EQUALS ONE IN SUBROUTINE STAT',MAN11940
     */,1X,'STANDARD DEVIATION, SKEW, AND KURTOSIS ARE INDETERMINATE')  MAN11950
      RETURN                                                            MAN11960
5     WRITE(6,300)                                                      MAN11970
300   FORMAT('0','***WARNING***',/,1X,'STANDARD DEVIATION IS ZERO IN ', MAN11980
     *'SUBROUTINE STAT',/,1X,'SKEW AND KURTOSIS COEFFICIENTS ARE ',     MAN11990
     *'INDETERMINATE')                                                  MAN12000
      RETURN                                                            MAN12010
      END                                                               MAN12020
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